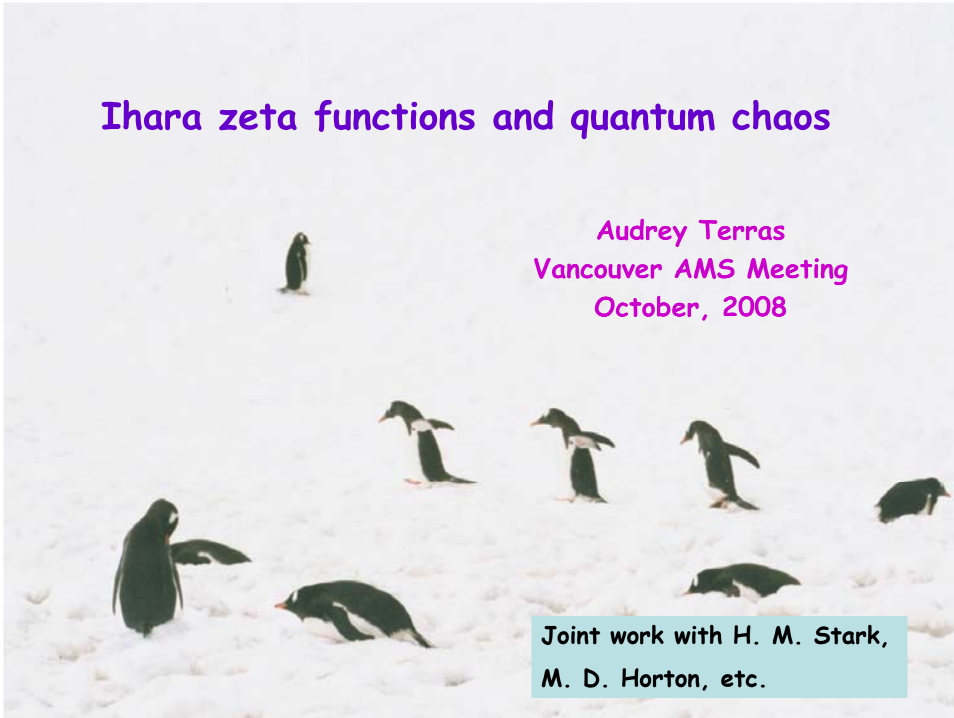
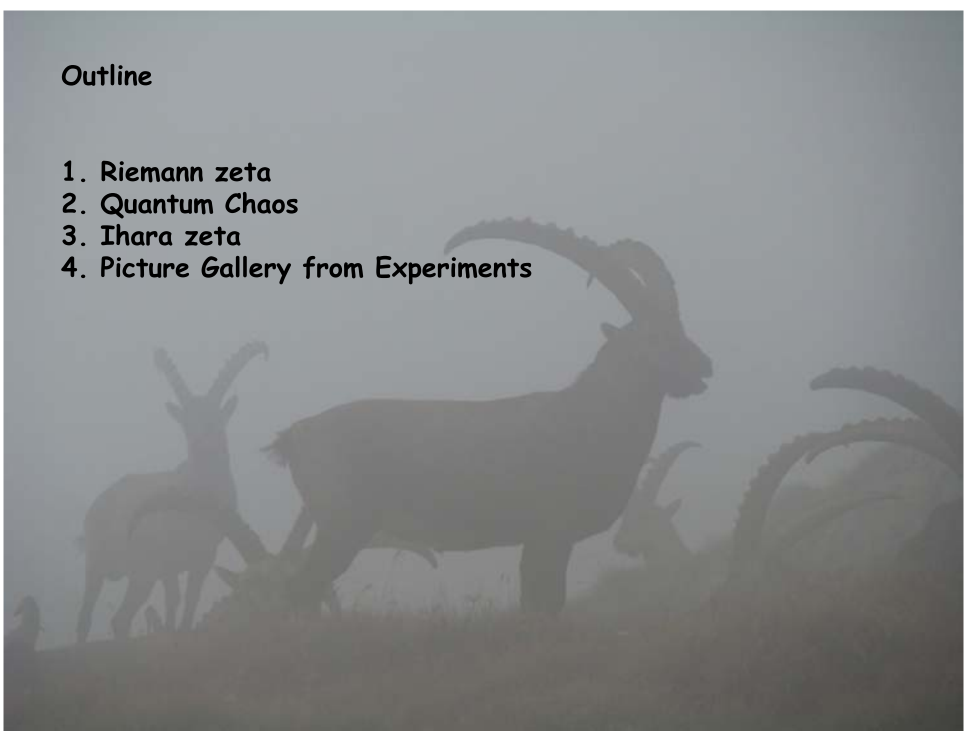

Ihara zeta functions and quantum chaos
Audrey Terras
Vancouver AMS Meeting
October, 2008
Joint work with H. M. Stark, M. D. Horton, etc.


Outline
1. Riemann zeta
2. Quantum Chaos
3. Ihara zeta
4. Picture Gallery from Experiments

Riemann ζ

The **Riemann zeta function** for Re(s)>1

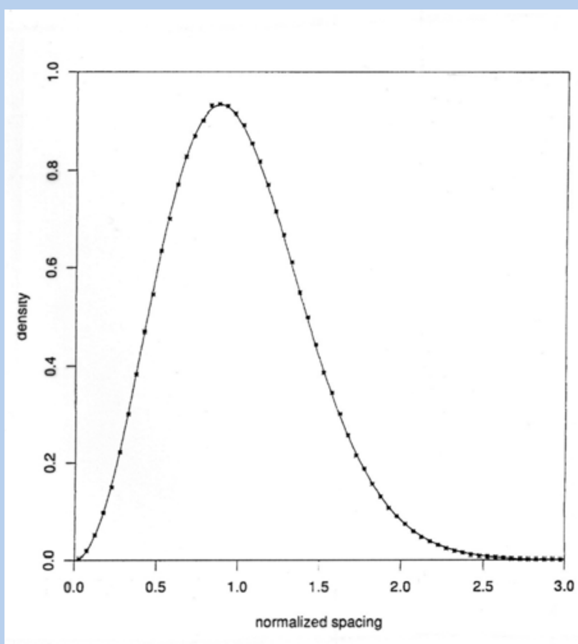
$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\prod_{p=prime}\left(1-p^{-s}\right)^{-1}.$$

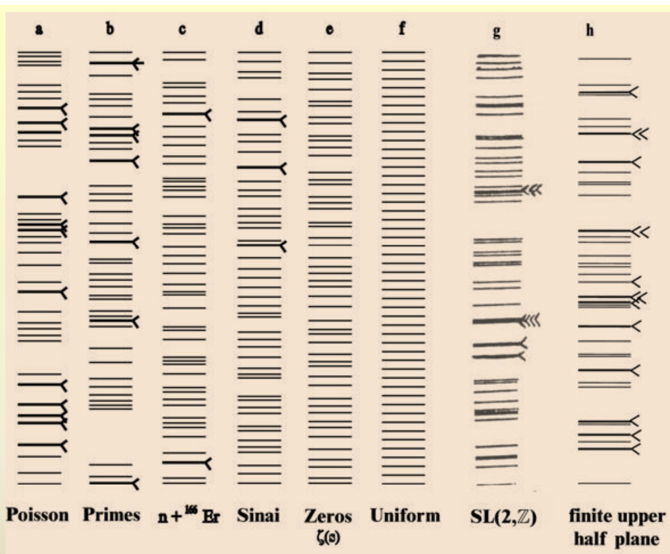
- Riemann extended to all complex s with pole at s=1 with a functional equation relating value at s and 1-s
- Riemann hypothesis
- primes <-> zeros of zeta

There are many other zetas:
Dedekind, primes -> prime ideals in number field
Selberg, primes->primitive closed geodesics in a manifold
Ihara, primes-> closed geodesics in graph

Odlyzko's Comparison of Spacings of Imaginary Parts of 1 million Zeros of Zeta near $2*10^{20}$ zero and Eigenvalues of Random Hermitian Matrix (GUE distribution).

See
B. Cipra, *What's Happening in the Mathematical Sciences, 1998-1999, A.M.S., 1999.*





From O. Bohigas and M.-J. Giannoni, Chaotic motion and random matrix theories, *Lecture Notes in Physics*, 209, Springer-Verlag, Berlin, 1984:

Arrows mean lines are too close to distinguish.

O. Bohigas and M.-J. Giannoni, Chaotic motion and random matrix theories, *Lecture Notes in Physics*, 209, Springer-Verlag, Berlin, 1984: "The question now is to discover the stochastic laws governing sequences having very different origins, as illustrated in" the Figure, each column with 50 levels ..." Note that the spectra have been rescaled to the same vertical axis from 0 to 49.

(a) Poisson spectrum, i.e., of a random variable with spacings of probability density e^{-x}.
(b) primes between 7791097 and 7791877.
(c) resonance energies of compound nucleus observed in the reaction n+166Er.
(d) from eigenvalues corresponding to transverse vibrations of a membrane whose boundary is the Sinai billiard which is a square with a circular hole cut out centered at the center of the square.
(e) the positive imaginary parts of zeros of the Riemann zeta function (from the 1551th to the 1600th zero).
(f) is equally spaced - the picket fence or uniform distribution.
(g) from P. Sarnak, Arithmetic quantum chaos, *Israel Math. Conf. Proc.*, 8 (1995), (published by Amer. Math. Soc.) : eigenvalues of the Poincaré Laplacian on the fundamental domain of the modular group SL(2,$\mathbb{Z}$), 2 × 2 integer matrices of determinant 1.
(h) spectrum of a finite upper half plane graph for p=53 (a = δ = 2), without multiplicity (see my book *Fourier Analysis on Finite Groups*)

Quantum Chaos (Chaology) is the study of the statistics of energy levels of quantum mechanical (& other) systems.

In the 1950's Wigner considered modeling the Schrödinger equation with the eigenvalue equation for a finite matrix. He obtained the histogram of the spectra of 197 real 20x20 symmetric matrices with normally distributed entries - close to a semicircle.

He found that such distributions are not similar to those observed in nature - although the semi-circle distribution is observed in number theory & graph theory.

Then he began to look at spacings of eigenvalues.

This lead to random matrix theory and work of Gaudin, Mehta, Dyson, etc.
See Mehta's book *Random Matrices*.
Bohigas, Quantum Chaos, *Nuclear Physics A*, 751 (2005)
Rudnick, Notices AMS, Jan., 2008

Wigner surmise for spacings of spectra of random symmetric real matrices

This means that you arrange the eigenvalues) E_i in decreasing order: $E_1 \geq E_2 \geq \cdot \quad \cdot \quad \cdot \geq E_n$. Assume that the eigenvalues are normalized so that the mean of the level spacings $|E_i - E_{i+1}|$ is 1.

Wigner's Surmise from 1957 says the level (eigenvalue) spacing histogram for the eigenvalues of a large random normal symmetric real matrix is ≈ the graph of the function $\frac{1}{2}\pi x \exp(-\pi x^2/4)$, if the mean spacing is 1.

In 1960, Gaudin and Mehta found the correct distribution function which is close to Wigner's. The correct graph is called GOE. Note the level repulsion indicated by the vanishing of the function at the origin.

The Poisson spacing density e^{-x}. No level repulsion there.

A reference is Mehta, *Random Matrices*.

The dichotomy

RMT spacings (GOE etc) $\frac{1}{2}\pi x \exp(-\pi x^2/4)$	Poisson spacings $\exp(-x)$
quantum spectra of system with chaos in classical counterpart Bohigas Giannoni Schmit Conjecture, 1984	energy levels of quantum system with integrable system for classical counterpart Berry Tabor Gutzwiller Conjecture, 1977
eigenvalues of Laplacian for non-arithmetic manifold	eigenvalues of Laplacian for arithmetic manifold; e.g. $H/SL(2,\mathbb{Z})$
zeros Riemann zeta	
poles Ihara zeta for random graph or random graph cover	poles Ihara zeta for covering with abelian Galois group

Sarnak invented the term "arithmetical quantum chaos" to describe the 2nd row of our table. See the book of Katz and Sarnak for some proved results about zetas of curves over finite fields.

See Rudnick "What is quantum chaos?" Notices AMS, Jan. 2008 for definitions of some of these things in the context of billiards. None of these conjectures is proved as far as I know.

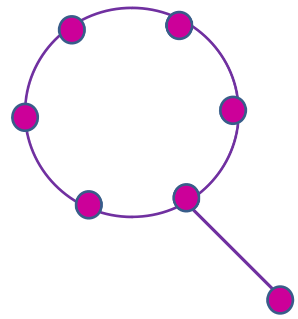
Finite Graphs

We want to investigate spectral statistics for zeta functions (equivalently matrices) coming from finite undirected connected graphs.

Usually we assume:

graph is not a cycle or
a cycle with degree 1
vertices

A **Bad Graph** is

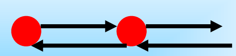


It is OK to be irregular (vertices do not all have same degree) and to have multiple edges and loops.
You can add physics to it by making the graph a quantum graph or just putting non-negative weights on the edges. This drastically changes the locations of poles.

Primes in Graphs

are equivalence classes [C] of closed backtrackless tailless primitive paths C
(geodesics in graph; i.e., minimize distance = # edges in path)

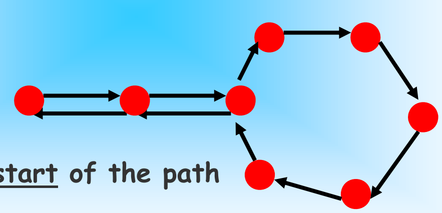
DEFINITIONS backtrack



equivalence class: change starting point

tail

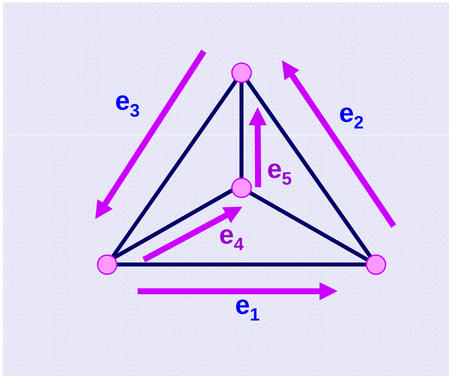
α



Here α is the <u>start</u> of the path

non-primitive: go around path more than once

EXAMPLES of Primes in a Graph



$[C] = [e_1 e_2 e_3]$

$[D]=[e_4 e_5 e_3]$

$[E]=[e_1 e_2 e_3 e_4 e_5 e_3]$

$\nu(C)$ = # edges in C

$\nu(C)=3$, $\nu(D)=3$, $\nu(E)=6$

E=CD
another prime $[C^n D]$, n=2,3,4, …
infinitely many primes

Ihara Zeta Function

Definition

$$\zeta(u,X)=\prod_{\substack{[C]\\ \text{prime}}}\left(1-u^{\nu(C)}\right)^{-1}$$

for u complex, |u| small

Theorem

$$\zeta(u,X)^{-1}=(1-u^2)^{r-1}\det(I-Au+Qu^2)$$

A=adjacency matrix, |V|x|V| matrix of 0s and 1s with i,j entry 1 iff vertex i adjacent to vertex j
Q +I =D= diagonal matrix of degrees of vertices, degree vertex = # edges coming out of vertex
I=identity matrix,
r=rank fundamental group = |E|-|V|+1

For q+1 – regular graph, $u=q^{-s}$ makes

Ihara zeta more like Riemann zeta.

$f(s)=\zeta(q^{-s})$ has a functional equation relating f(s) and f(1-s).

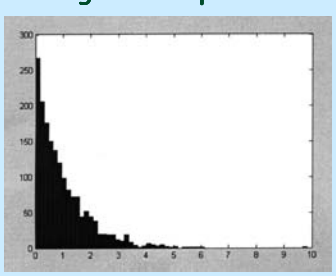
Riemann Hypothesis (RH)

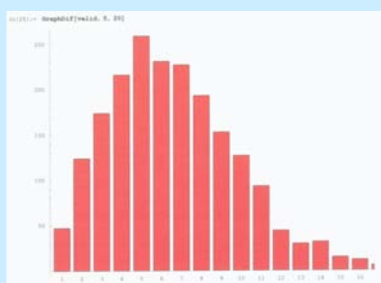
says $\zeta(q^{-s})$ has no poles with $0<\mathrm{Re}\,s<1$ unless $\mathrm{Re}\,s = \frac{1}{2}$.

RH means graph is Ramanujan i.e., non-trivial spectrum of adjacency matrix is contained in the spectrum for the universal covering tree which is the interval $(-2\sqrt{q}, 2\sqrt{q})$ [see Lubotzky, Phillips & Sarnak, *Combinatorica*, 8 (1988)].

and thus a good expander

Derek Newland's Experiments on Spacings of Poles of Zeta in Regular Graph Case





Spacings of Zeros of Ihara Zetas of Regular Graphs

On the Left the Graph is a Finite Euclidean Graph $Euc_{1999}(2,1)$ as in Chapter 5 of my book *Fourier Analysis on Finite Groups and Applications.* It is a Cayley graph for a finite abelian group.

On the Right is a Random Regular Graph as given by Mathematica with 2000 vertices and degree 71.

Moral: Cayley graph of abelian group spacings look Poisson

random graph spacings look GOE

What is the the RH for an irregular graph?

For irregular graph, natural change of variables is $u=R^s$, where

R = radius of convergence of Dirichlet series for Ihara zeta.

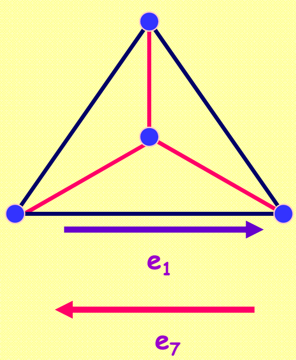
Note: R is closest pole of zeta to 0.

There is no functional equation.

The critical strip is $0 \leq \mathrm{Re}(s) \leq 1$. We only consider the right half.

Graph theory RH: $\zeta(u)$ is pole free in $R < |u| < \sqrt{R}$.

Labeling Edges of Graphs



X = finite connected (not-necessarily regular graph)
Orient the m edges.
Label them as follows.
Here the inverse edge has opposite orientation.

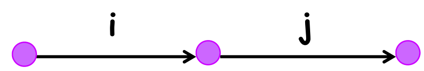
$e_1, e_2, \ldots, e_m,$

$e_{m+1}=(e_1)^{-1}, \ldots, e_{2m}=(e_m)^{-1}$

With this labeling, we have the properties of the edge matrix defined on the next slide.

The Edge Matrix W

Define W to be the $2|E|\times 2|E|$ matrix with i j entry 1 if edge i feeds into edge j, (end vertex of i is start vertex of j) provided that $j \neq$ the inverse of i, otherwise the i j entry is 0.



Theorem. $\zeta(u,X)^{-1}=\det(I-Wu)$.

Corollary. The poles of Ihara zeta are the reciprocals of the eigenvalues of W.

The pole R of zeta is:

R=1/Perron-Frobenius eigenvalue of W; i.e., the largest eigenvalue which has to be positive real.

Properties of W

1) $W=\begin{pmatrix} A & B \\ C & A^T \end{pmatrix}$, B and C symmetric & diagonal entries are 0

2) Row sums of entries are q_j+1=degree jth vertex
3) Singular Values (square roots eigenvalues of WW^T) are $\{q_1,\ldots,q_n,1,\ldots,1\}$.
4) $(I+W)^{2|E|-1}$ has all positive entries, if $2 \leq r$
r=rank fundamental group.
So we can apply Perron-Frobenius theorem to W.

Poles Ihara Zeta

are in region

$$q^{-1} \leq R \leq |u| \leq 1,$$

q+1=maximum degree of vertices of X.

So eigenvalues of W being reciprocals of poles are outside unit circle and inside circle of radius q.

Theorem of Kotani and Sunada

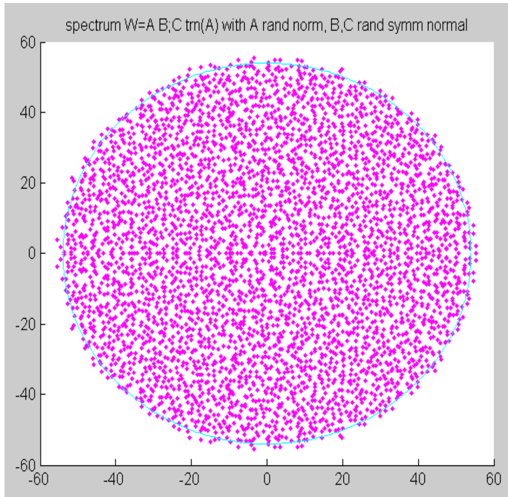
1. If p+1=min vertex degree, and q+1=maximum vertex degree, non-real poles u of zeta satisfy

$$\frac{1}{\sqrt{q}} \le |u| \le \frac{1}{\sqrt{p}}$$

2. Poles symmetric under rotation by $2\pi/\Delta$, where Δ = g.c.d. lengths of primes in graph

Kotani & Sunada, *J. Math. Soc. U. Tokyo*, 7 (2000)

Spectrum of Random Matrix with Properties of W-matrix



$$W = \begin{pmatrix} A & B \\ C & A^T \end{pmatrix}$$

entries of W are non-negative from normal distribution

B and C symmetric diagonal entries are 0

Girko circle law for real matrices with circle of radius ½(1+√2) √n

symmetry about real axis

Can view W as edge matrix for a weighted graph

We used Matlab command randn(1000) to get A,B,C matrices with random normally distributed entries mean 0 std dev 1

The Girko circle law says that the spectrum of an nxn real non-symmetric matrix with entries independent and taken from a standard normal distribution should become uniformly distributed in a circle of radius √n as n goes to infinity. See Bai, *Ann. Prob.*, 1997.

Here although W is not symmetric, the nearest neighbor spacing (i.e., histogram of minimum distances between eigenvalues) is also of interest. The Wigner surmise for a complex non-symmetric large matrix is:

$$4\Gamma\left(\frac{5}{4}\right)^4 s^3 e^{-\Gamma\left(\frac{5}{4}\right)^4 s^4}$$

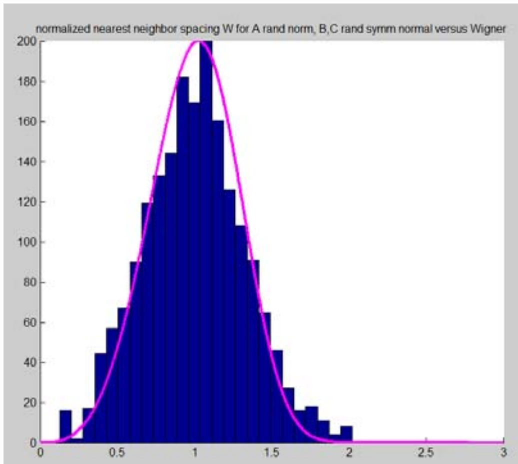
References:

P. LeBoef, Random matrices, random polynomials, and Coulomb systems.
J. Ginibre, *J. Math. Phys.* 6, 440 (1965).

Mehta, *Random Matrices*, Chapter 15.

Nearest Neighbor Spacings vs Wigner surmise of Ginibre

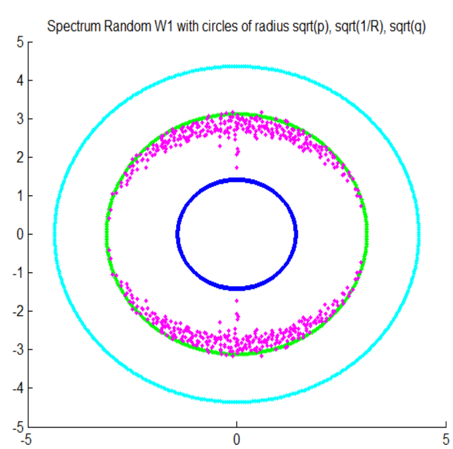
$p_w(s) = 4\ \Gamma(5/4)^4\ s^3 \exp\left(-\Gamma(5/4)^4 s^4\right)$



See P. Leboef, "Random matrices, random polynomials and Coulomb systems," ArXiv, Nov. 15, 1999.

Mehta, *Random Matrices*, Chapter 15

Matlab Experiments with Eigenvalues of "Random" W matrix of an Irregular Graph - Reciprocals of Poles of Zeta



Circles have radii

$\sqrt{p}$ blue

$1/\sqrt{R}$ green

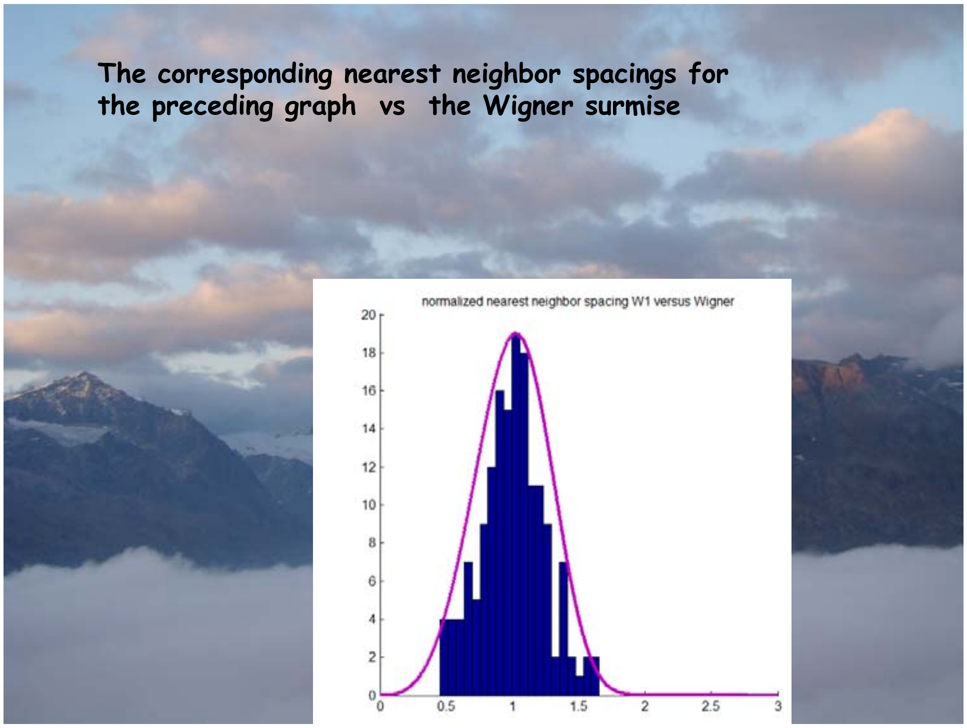
$\sqrt{q}$ turquoise

RH approximately true region 2 dimensional but not even an annulus

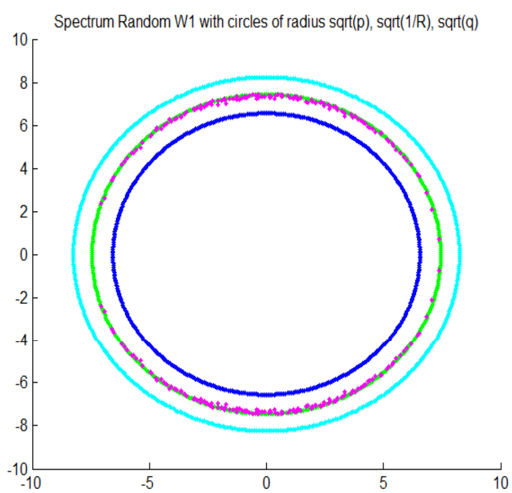
Looks very similar to the regions obtained for random covers of a small base graph.

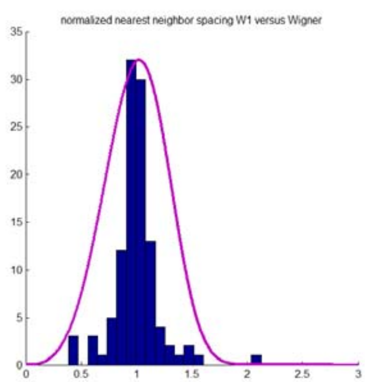
probability of edge ≈.036

RH ≈ true



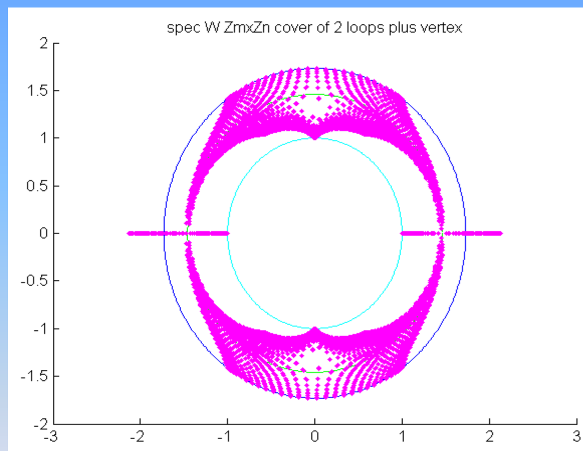
Matlab Experiment Random Graph with High Probability of an Edge Between Vertices (Edge Probability ≈ .74)

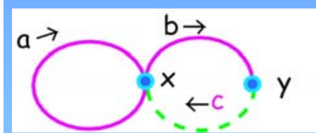




RH ≈ true

Spectrum W for a $\mathbb{Z}_{61}\times\mathbb{Z}_{65}$-Cover of 2 Loops + Extra Vertex are pink dots

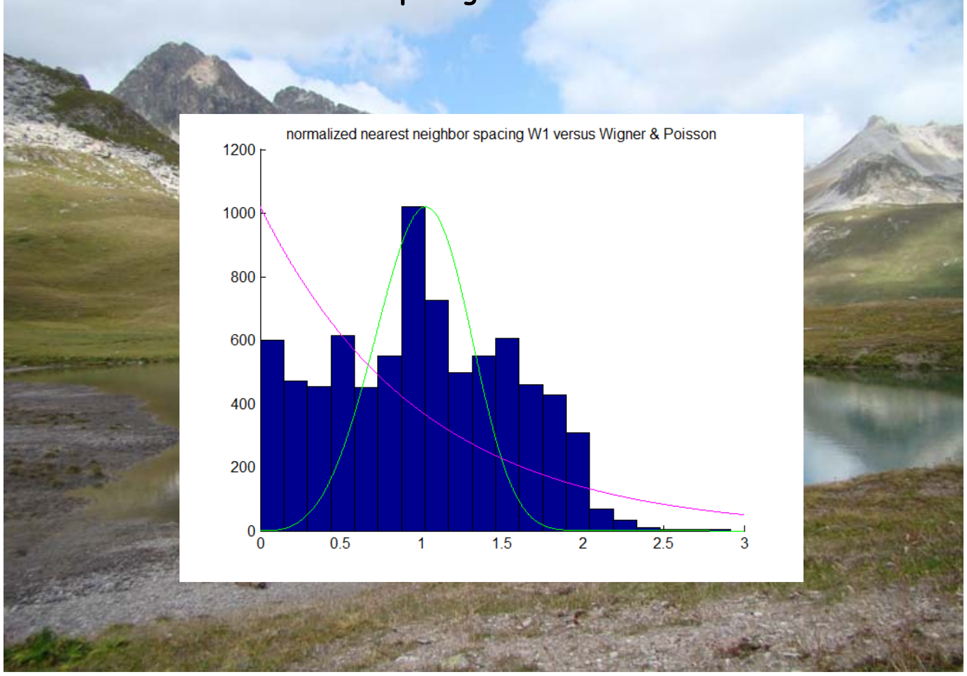




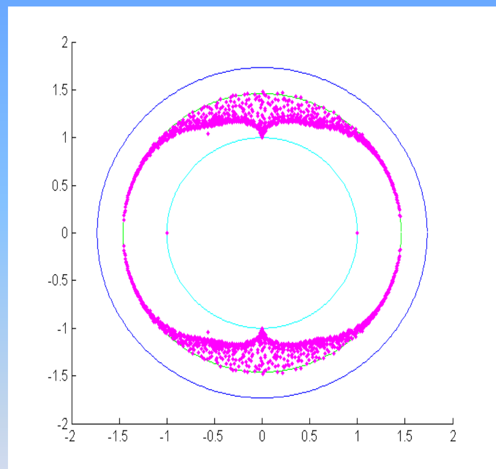
Circles Centers (0,0); Radii: $3^{-1/2}$, $R^{1/2}$,1; R ≅.47

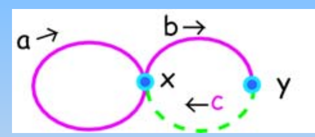
RH very False: Lots of Pink outside green circle

Level Spacing for the Previous W Matrix



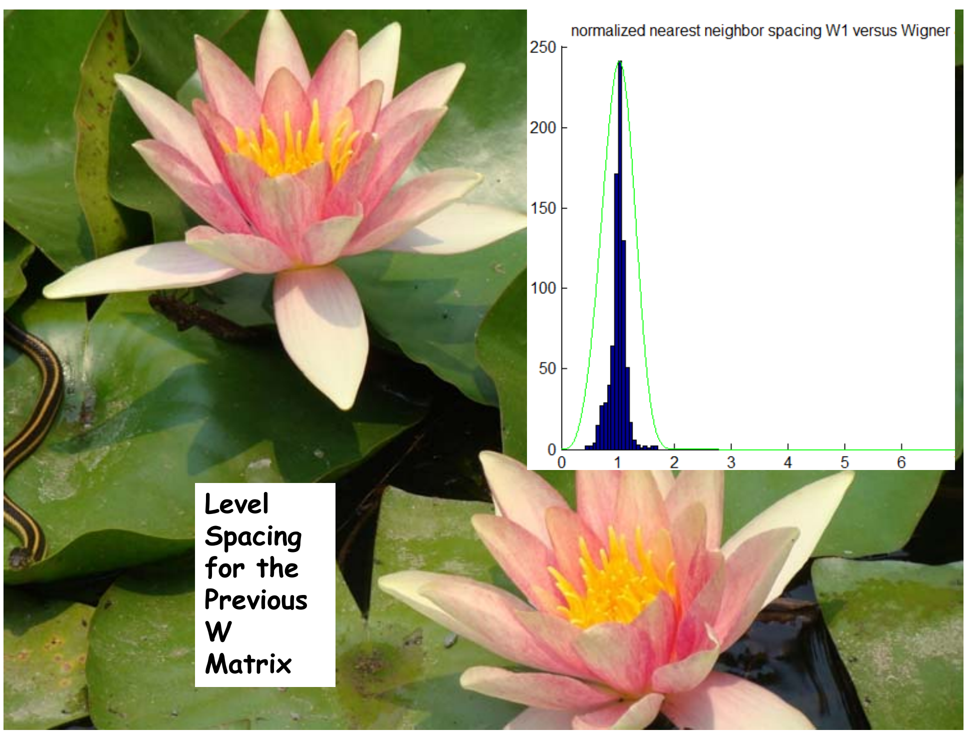
Spectrum W Matrix for Random 455 cover of 2 loops plus vertex





Z is random 455 cover of 2 loops plus vertex graph in picture.

The pink dots are at Spectrum W. Circles have radii $\sqrt{q}$, $1/\sqrt{R}$, $\sqrt{p}$, with $q=3$, $p=1$, $R \cong .4694$. RH approximately True.



References: 3 papers with Harold Stark in *Advances in Math.*

- Papers with Matthew Horton & Harold Stark in

  Quantum Graphs and Their Applications, *Contemporary Mathematics*, Vol. 415, AMS, Providence, RI 2006.

  Analysis on Graphs & its Applications, *Proc. Symp. Pure Math.*, Vol. 77, AMS, Providence, RI 2008

- See my draft of a book:

  www.math.ucsd.edu/~aterras/newbook.pdf

- There was a graph zetas special session of the Jan., 2008, AMS meeting – many interesting papers some on my website.

- For work on directed graphs, see Matthew Horton, Ihara zeta functions of digraphs, *Linear Algebra and its Applications,* 425 (2007) 130–142.

Work of Friedman, Angel & Hoory

The non-backtracking spectrum of the universal cover of a graph, preprint on Joel Friedman's website at UBC.

W is their non-backtracking adjacency matrix. They give a method to find is spectrum of the corresponding operator on the universal covering tree of the base graph X. One small example is drawn (K4-edge)

Irregular Graph Analog of Alon Conjecture which was for regular graphs (proved in a paper on Friedman's website): new (i.e., not from X) spectrum of W-matrix for random n-sheeted covering (lift) Y of X should approach region D as $n \to \infty$.

$R^{-1/2}$ is the spectral radius of the W-operator on the universal cover of X.

So approximate RH for covers of fixed base graph is contained in this irregular Alon conjecture of Friedman, Angel & Hoory


The End