**Solutions to Math 142 B Exam 1**

1) a) A partition $P$ of $[a,b]$ is a set of points $x_i$ such that

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

An example is the regular partition $P_n$ of $[0,1]$, given by

$$P_n = \{x_i = \frac{i}{n}, i = 0, 1, ..., n\}.$$

b) A refinement $Q$ of partition $P$ of $[a,b]$ is a partition $Q$ of $[a,b]$ such that $P \subset Q$. As an example, note that if we use the notation of part a) $P_{2n}$ is a refinement of $P_n$.

2) a) Suppose that $f : [a,b] \to \mathbb{R}$ is bounded. Then $f$ is integrable on $[a,b]$ if and only if there is a sequence $\{P_n\}$ of partitions of $[a,b]$ such that

$$\lim_{n\to\infty} (U(f,P_n) - L(f,P_n)) = 0.$$

Moreover, for any such sequence of partitions,

$$\int_a^b f = \lim_{n\to\infty} U(f,P_n) = \lim_{n\to\infty} L(f,P_n).$$

b) Let $F : [a,b] \to \mathbb{R}$ be continous on the closed interval $[a,b]$ and differentiable on the open interval $(a,b)$. Suppose, in addition, that the derivative $F' : (a,b) \to \mathbb{R}$ is continuous and bounded (thus integrable on $[a,b]$). Then

$$\int_a^b F' = F(b) - F(a).$$

3) a) Suppose that $f$ is monotone decreasing on $[a,b]$. Let

$$P_n = \{x_0, x_1, ..., x_n\}$$

be a regular partition of $[a,b]$; *i.e.*,

$$x_i = a + \frac{i(b-a)}{n}, i = 0, 1, 2, ..., n.$$

By the fact that $f$ is monotone decreasing, we see that

$$\begin{aligned} M_i &= \sup\{f(x)|x \in [x_{i-1}, x_i]\} = f(x_{i-1}) \\ m_i &= \inf\{f(x)|x \in [x_{x-1}, x_i]\} = f(x_i). \end{aligned}$$

Thus

$$\begin{aligned} & U(f, P_n) - L(f, P_n) \\ = & \sum_{i=0}^{n} f(x_{i-1})\frac{b-a}{n} - \sum_{i=0}^{n} f(x_i)\frac{b-a}{n} \\ = & \frac{b-a}{n}\sum_{i=0}^{n}(f(x_{i-1}) - f(x_i)) \\ = & \frac{b-a}{n}(f(a) - f(b)). \end{aligned}$$

For the last equality, we used the fact that the preceding sum is telescoping.

b) It follows that as $n$ approaches infinity, $\frac{b-a}{n}(f(a) - f(b))$ approaches 0, since $(b-a)(f(a) - f(b))$ is a constant and $\frac{1}{n} \to 0$, as $n \to \infty$. Thus $f$ is integrable, by the Archimedes-Riemann Theorem.

4) a) **False.** Step functions are integrable but not continuous.

b) **True.** The value of a function at a particular point does not affect the integral. Let $c_n = a + 1/n$. Note that $|(g - f)(x)| \leq 1$, for all $x \in [a, b]$. Then $\int_a^b (g - f) = \int_a^{c_n} (g - f) + \int_{c_n}^b 0 = \int_a^{c_n} (g - f) \leq \frac{1}{n} \to 0$, as $n \to \infty$.

5) Let $P_n = \{x_0, x_1, ..., x_n\}$ be a partition of $[a, b]$. We apply the mean value theorem (text, p. 103) to see that there is a point $c_i$ in $(x_{i-1}, x_i)$ so that

$$\frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}} = F'(c_i).$$

Then set

$$\begin{aligned} m_i &= \inf\{F'(x)|x \in [x_{i-1}, x_i]\} \\ M_i &= \sup\{F'(x)|x \in [x_{i-1}, x_i]\}. \end{aligned}$$

It follows that $m_i \leq F'(c_i) \leq M_i$. Thus

$$L(F', P) \leq \sum_{i=1}^{n} F'(c_i)(x_i - x_{i-1}) \leq U(F', P).$$

The mean value formula $\frac{F(x_i)-F(x_{i-1})}{x_i-x_{i-1}} = F'(c_i)$ implies that

$$F(x_i) - F(x_{i-1}) = F'(c_i)(x_i - x_{i-1}).$$

If we substitute this into the middle sum above, we see that

$$L(F', P_n) \leq \sum_{i=1}^{n} (F(x_i) - F(x_{i-1})) \leq U(F', P_n).$$

But the middle sum here is telescoping and thus equals $F(b) - F(a)$. We know that our upper and lower sums both approach the same limit since $F^{'}$ is integrable. From the Archimedes-Riemann theorem, we know that

$$\lim_{n\to\infty} L(F', P_n) = \int\limits_a^b F^{'} = \lim_{n\to\infty} U(F', P_n).$$

Since limits preserve $\leq$, we see that

$$\int\limits_a^b F^{'} \leq F(b) - F(a) \leq \int\limits_a^b F^{'}.$$

It follows that

$$\int\limits_a^b F^{'} = F(b) - F(a).$$