Midterm 1 – Math 20D (blue form)

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1) Solve the separable ODE $\dfrac{dy}{dx} = \dfrac{x^2}{y}$ with initial condition $y(3) = 2$.
Find the interval $a \le x \le b$ where the solution $y(x)$ is defined.

SOLUTION: We first separate to get

$$y\,dy = x^2\,dx, \quad \text{then integrating gives} \quad \frac{1}{2}y^2 = \frac{1}{3}x^3 + C.$$

We now solve for C by using our initial condition

$$\frac{1}{2}2^2 = \frac{1}{3}3^3 + C, \quad \text{or} \quad 2 = 9 + C, \quad \text{so} \quad C = -7.$$

We then substitute in for C and rearrange to get

$$y^2 = \frac{2}{3}x^3 - 14, \quad \text{so} \quad y = \pm\sqrt{\frac{2}{3}x^3 - 14}.$$

But we need to have $y(3) = 2$ so we should choose the "+" square root, i.e.,

$$\boxed{y = \sqrt{\frac{2}{3}x^3 - 14}.}$$

To find the interval where the solution is defined we see that our only problem is that we have a square root and so we need the term inside the square root to be positive, i.e.,

$$\frac{2}{3}x^3 - 14 \ge 0 \quad \text{or} \quad x^3 \ge 21 \quad \text{or} \quad \boxed{x \ge \sqrt[3]{21}.}$$

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2) The population $P(t)$ of a country at time t increases at a rate proportional to the number of people present at time t. Assume the proportionality factor is 4% per year.

a) Set up the ODE for this situation.

SOLUTION: $\boxed{\dfrac{dP}{dt} = 0.04\,P.}$

b) Find the formula for $P(t)$.

SOLUTION: We can separate it so that

$$\frac{dP}{P} = 0.04\,dt, \quad \text{now integrating gives} \quad \ln P = 0.04t + A.$$

Putting both sides into the exponential function (and bringing the constant down in front) will then give us

$$e^{\ln P} = e^{0.04t+A} \quad \text{or} \quad \boxed{P(t) = Be^{0.04t}.}$$

c) How many years would it take for the population of the country to double? Don't compute the decimal approximation !

SOLUTION: The population at time 0 is $P(0) = B$, so the population will be doubled at a time τ when $P(\tau) = 2B$. So we need

$$2B = Be^{0.04\tau} \quad \text{or} \quad 2 = e^{0.04\tau} \quad \text{or} \quad \ln 2 = 0.04\tau.$$

So the time τ it takes for the population to double is

$$\tau = \boxed{\frac{\ln 2}{0.04}.}$$

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3) True - False. Tell whether the following statements are true or false.
Give a reason for your answer.

a) The ODE $(3x^2 - y)dx - xdy = 0$ is exact.

SOLUTION: This is in the form $M\,dx + N\,dy = 0$ with $M = 3x^2 - y$ and $N = -x$. To be exact we need

$$\frac{\partial M}{\partial y} = -1 \text{ to be equal to } \frac{\partial M}{\partial x} = -1.$$

Since it is easy to check that they are equal then the equation is exact making the statement $\boxed{\text{true.}}$

b) The initial value problem $\dfrac{dy}{dx} = y^2, y(0) = 1$ has a solution for $x > 1$.

SOLUTION: From the book we know that locally this function has a solution to the initial value problem. Here the question is wondering if this solution can be extended to work for $x > 1$. First we find the solution, which is straightforward since the equation is separable.

$$\frac{dy}{y^2} = dx \text{ which on integrating becomes } -\frac{1}{y} = x + C.$$

The initial condition $y(0) = 1$ becomes $C = -1$. So we have $-(1/y) = x - 1$ or

$$y = \frac{1}{1-x}.$$

This function is not continuous at 1 (indeed it blows up at a vertical asymptote). The solution to an initial value differential equation cannot be passed through the vertical asymptote, and so the solution does not work for $x > 1$. In other words the statement is $\boxed{\text{false.}}$

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4) a) Find a fundamental set of solutions for $y'' - 3y' + 2y = 0$.

SOLUTION: This is a constant coefficient linear homogeneous differential equation and so we can translate this into solving a quadratic for r, i.e., solve $r^2 - 3r + 2 = 0$. This factors as $(r-1)(r-2) = 0$, so the roots are 1 and 2. Each root now translates back into a solution so that a fundamental set of solutions is

$$\boxed{\{y_1(t) = e^t, y_2(t) = e^{2t}\}.}$$

b) Solve the initial value problem:

$$y'' - 4y' + 3y = 0, \quad y(0) = 1, \quad y'(0) = 3.$$

SOLUTION: As in the previous part we first solve for r in the equation $r^2 - 4r + 3 = 0$. This factors as $(r-1)(r-3) = 0$ so that the roots are 1 and 3. In particular we know the general form of the solution is

$$y = Ce^t + De^{3t}. \qquad \text{(Note: } y' = Ce^t + 3De^{3t}.\text{)}$$

We now solve for C and D using the initial conditions.

$$\begin{aligned} y(0) &= C + D = 1 \\ y'(0) &= C + 3D = 3 \end{aligned}$$

Subtracting the first equation from the second equation the C's cancel and we have $2D = 2$, so $D = 1$. Putting this in for D we have $C + 1 = 1$ so $C = 0$. Finally substituting the values of C and D in we find the solution to the initial value problem is

$$\boxed{y = e^{3t}.}$$