1) Suppose that R, S are rings, A is a subring of R, B is an ideal of S. Let $\phi: R \rightarrow S$ be a ring homomorphism.
   a) Show that for all $r$ in $R$ and all $n$ in $\mathbb{Z}^+$, we have $\phi(nr) = n \phi(r)$ and $\phi(r^n) = \phi(r)^n$.
   b) Show that $\phi(A)$ is a subring of $S$. Here the image of $A$ is $\phi(A) = \{ \phi(a) | a \in A \}$.
   c) Show that if $A$ is an ideal in $R$ and $\phi(R) = S$, then $\phi(A)$ is an ideal of $S$.

2) Under the same hypotheses as in 1) prove:
   a) $\phi^{-1}(B)$ is an ideal of $R$. Here the inverse image of $B$ is $\phi^{-1}(B) = \{ a \in A | \phi(a) \in B \}$. We are not assuming the inverse function of $\phi$ exists.
   b) If $R$ has an identity 1 for multiplication, $S \neq \{0\}$, and $\phi(R) = S$, then $\phi(1)$ is the identity for multiplication in $S$.
   b) If $\phi$ is a isomorphism of $R$ onto $S$, then $\phi^{-1}$ is an isomorphism of $S$ onto $R$.

3) a) Show that $\phi: \mathbb{Z}_5 \rightarrow \mathbb{Z}_{10}$ defined by $\phi(x) = 5x$ does not preserve addition.
   b) Show that $\phi: \mathbb{Z}_4 \rightarrow \mathbb{Z}_{12}$ defined by $\phi(x) = 3x$ does not preserve multiplication.
   c) Show that every homomorphism $\phi: \mathbb{Z}_n \rightarrow \mathbb{Z}_n$ has the form $\phi(x) = ax$ for some fixed $a$ in $\mathbb{Z}_n$ with $a^2 = a$.

4) a) Show that the ring of complex numbers $\mathbb{C}$ is isomorphic to the factor ring $\mathbb{R}[x]/<x^2+1>$. Here $\mathbb{R}[x]$ is the ring of polynomials in 1 indeterminate $x$.
   b) Show that complex conjugation $\phi(a+ib) = a-ib$, for $a, b$ in $\mathbb{R}$ and $i^2 = -1$, defines a ring isomorphism from $\mathbb{C}$ onto $\mathbb{C}$.
   c) Show that $\mathbb{C}$ is not isomorphic to $\mathbb{R}$.
   d) Show that $\mathbb{C}$ is isomorphic to the ring $\left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} | a, b \in \mathbb{R} \right\}$.

5) a) Find all ring homomorphisms from the rationals $\mathbb{Q}$ to $\mathbb{Q}$.
   b) Show that the only ring isomorphism mapping the reals $\mathbb{R}$ onto $\mathbb{R}$ is the identity map.