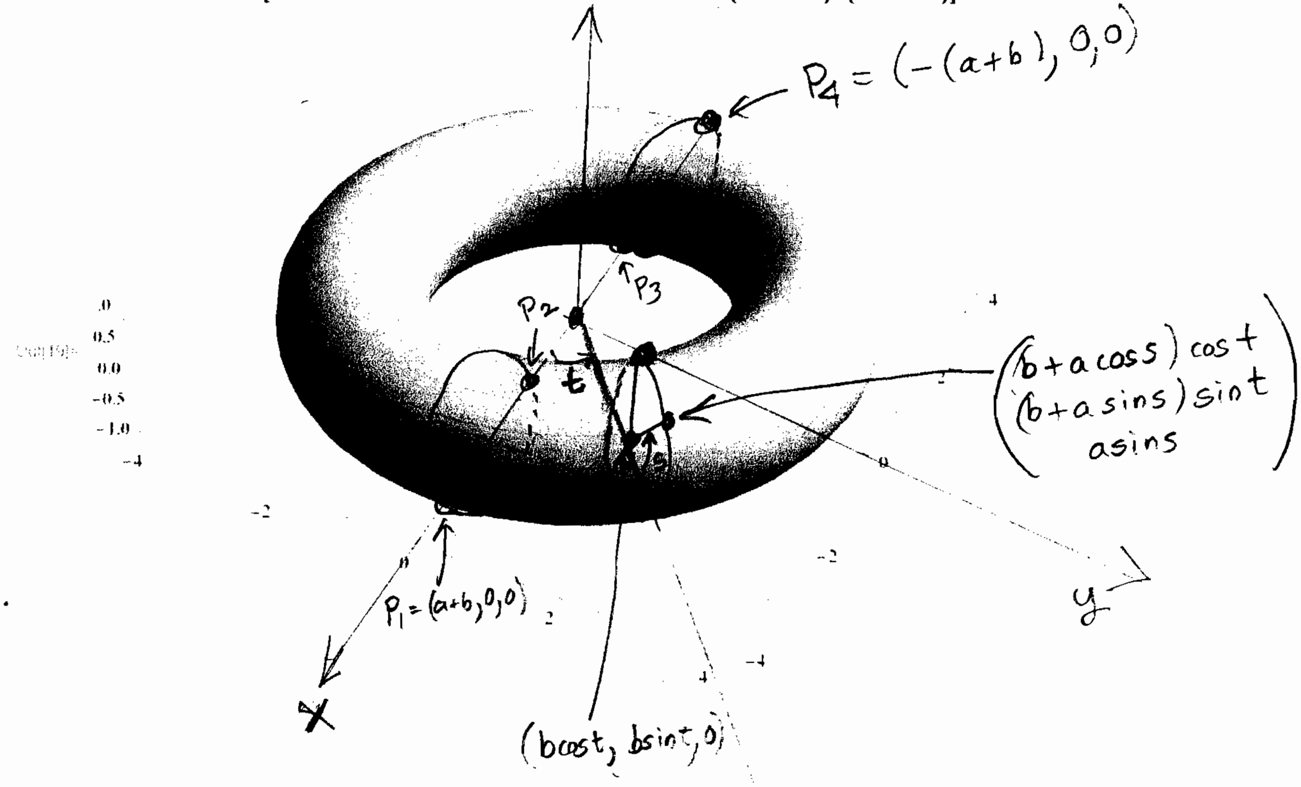
p.339 #12, Mathematica draws a torus

Homework 2 Solutions

```
ParametricPlot3D[{(3 + Cos[v]) Cos[u], (3 + Cos[v]) Sin[u], Sin[v]}, {u, 0, 2Pi}, {v, 0, 2Pi}]
```



b) $f_1'(s,t) = (-a \sin s \cos t, -(b + a\cos s)\sin t)$ Assume $0 \le s, \pi < 2\pi$

c) $\sin s \cos t = 0$, $(b + a\cos s)\sin t = 0$.

$\Rightarrow$ either $\sin s = 0$ & then $\cos s = \pm 1$ and $(b+a)\sin t = 0$ or $(b-a)\sin t = 0$ $\Big\} \Rightarrow \sin t = 0$

or $\cos t = 0$ & then $\sin t = \pm 1$ so $b + a\cos s = 0 \Rightarrow \cos s = \frac{-b}{a}$

But $|\frac{b}{a}| > 1 \Rightarrow$ not possible

So only 1st case occurs.

$(s,t) = (0,0), (0,\pi), (\pi, 0), (\pi,\pi)$

corresponding $f(s,t)$ are $P1 = (a+b, 0, 0)$, $P2 (b-a, 0, 0)$

$P3 = (-(b-a), 0, 0)$, $P4 (-(a+b), 0, 0)$

You can see from the figure that Clearly P1 yields a global max for the x-coordinate, P4 a global min

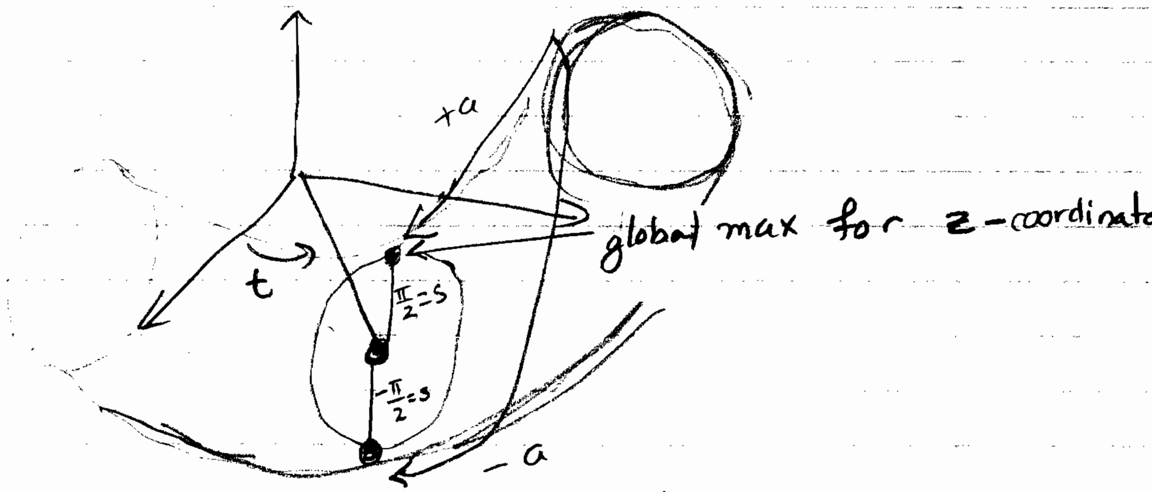
P2 & P3 are saddle points.

p. 239 #12 (cont'd)

b) $f_3'(s,t) = (\ a\cos s,\ 0) = 0 \iff \cos s = 0$

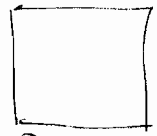
c) $\iff s = \frac{\pi}{2}, \frac{3\pi}{2}$, t arbitrary

$\Rightarrow f_1(s,t) = (b\cos t,\ b\sin t,\ \pm a)$



d) $$f'(s,t) = \begin{pmatrix} -a\sin s\cos t & -(b+a\cos s)\sin t \\ -a\sin s\sin t & (b+a\cos s)\cos t \\ a\cos s & 0 \end{pmatrix}$$

$\forall (s,t)$ one can see that $\exists$ a 2×2 minor of this matrix having determinant $\neq 0$ (next page)
Thus by homework 3 the mapping f provides a local 1-1, onto, homeomorphism/diffeo topologically identifying the torus & the square $[0,2\pi]^2$.
We can easily change variables to identify $[0,2\pi]^2$ & $[0,1]^2$ topologically also.

Why can we identify  and torus?

$$F'(t) = \begin{array}{cc} & \begin{matrix} \quad\frac{\partial}{\partial s} & \qquad\qquad\qquad \frac{\partial}{\partial t} \end{matrix} \\ \begin{matrix} f_1 \\ f_2 \\ f_3 \end{matrix} & \begin{pmatrix} -a\sin s\cos t & -(b+a\cos s)\sin t \\ -a\sin s\sin t & +(b+a\cos s)\cos t \\ a\cos s & 0 \end{pmatrix} \end{array}$$

$\forall t$ the rank of this matrix is 2

$$\det\begin{bmatrix} -a\sin s\cos t & -(b+a\cos s)\sin t \\ -a\sin s\sin t & +(b+a\cos s)\cos t \end{bmatrix} = -a\left\{\begin{matrix}\sin s\cos^2 t\,(b+a\cos s) \\ +\sin s\sin^2 t\,(b+a\cos s)\end{matrix}\right\} = -a\sin s\,(\underbrace{b+a\cos s}_{\neq 0})$$

$$\det\begin{bmatrix} -a\sin s\cos t & -(b+a\cos s)\sin t \\ a\cos s & 0 \end{bmatrix} = -a\cos s\sin t\,(\underbrace{b+a\cos s}_{\neq 0})$$

$$\det\begin{bmatrix} -a\sin s\sin t & (b+a\cos s)\cos t \\ a\cos s & 0 \end{bmatrix} = a\cos s\cos t\,(\underbrace{b+a\cos s}_{\neq 0})$$

They can't all vanish at once

$\sin s = 0 \Leftrightarrow \cos s \neq 0$

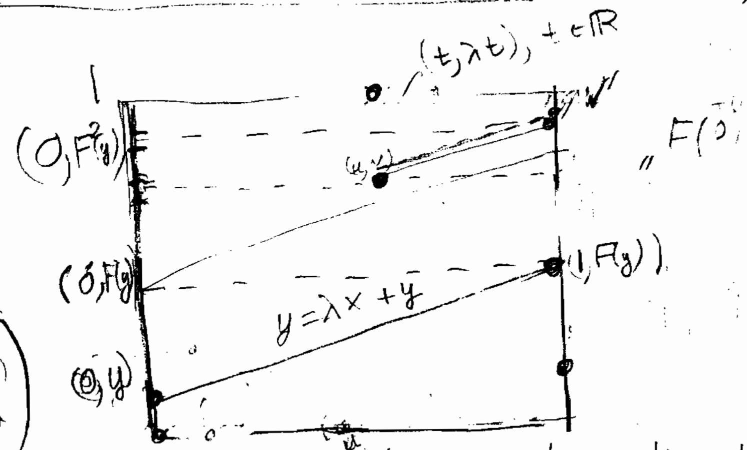
$\sin t = 0 \Leftrightarrow \cos t \neq 0$

This means f is 1-1 map from $[0,2\pi)^2$ onto $\mathbb{T}^2$.
So we can identify $[0,2\pi)^2$ and $\mathbb{T}^2$ topologically
f is continuous & diffentiable.

Next we think about <u>the densely wound line</u> (assuming irrational slope) <u>in the torus.</u>

Change variable $(s,t) = (2\pi x, 2\pi y)$
$0 \le x, y < 1$

λ irrational
$\Rightarrow$ every orbit in $\mathbb{T}^2$ is dense.
Look at orbit thru $(0,0)$ $\qquad (2\pi x, 2\pi\lambda x) \quad x \in [0,1)$



Orbit thru $(0,y)$ instead

$(1, F(y)) = 1^{st}$ point where line hits $x=1$
Compute F

orbit thru $(0,y)$ is $(t, \lambda t + y) \pmod{\mathbb{Z}^2}$

$$\begin{cases} t = 1 \\ \lambda t + y = F(y) \end{cases} \qquad F: \mathbb{T} \to \mathbb{T} \quad (1^{st} \text{ return map})$$

$\lambda < 1 \Rightarrow y = \lambda$ here

$$\Rightarrow \quad F(y) = \lambda + y \pmod{\mathbb{Z}}$$
$$\Rightarrow \quad F^n(y) = n\lambda + y \pmod{\mathbb{Z}}$$

To see denseness of $(t, \lambda t + y) \pmod{\mathbb{Z}^2}$, $t \in \mathbb{R}$
suffices to show

$$\{F^n(y), n \ge 0\} \text{ dense mod } \mathbb{Z}. \qquad \text{Why?}$$

Take any pt (u,v)
Follow line of slope λ to $x=1$
Can find element of A close to y-coordinate
Line thru this will be close to (u,v)

This is statement:
$A = \{n\lambda + m \mid n, m \in \mathbb{Z}\}$ dense in $\mathbb{R}$

(1) A is additive subgp of $\mathbb{R}$ (why: See (4))
(2) suffices to show 0 is an acc. pt. of A
(3) $\forall\, n \ge 0$ let $m_n \in \mathbb{Z}$ be s.t. $x_n = n\lambda + m_n \in [0,1) \cap A$
$\lambda \notin \mathbb{Q} \Rightarrow x_n$ all distict
$[0,1]$ compact $\Rightarrow \exists$ convergent subseq $\{x_{n_k}\}$
$\forall\, \varepsilon > 0 \;\exists\, x_{n_1}, x_{n_2}$ s.t. $|x_{n_1} - x_{n_2}| < \varepsilon$
But the $x_{n_1} - x_{n_2} \in A \cap (-\varepsilon, \varepsilon)$

(4) 0 an acc pt of $A \Rightarrow$ given $u \in \mathbb{R}$
$\Rightarrow u =$ acc pt of A

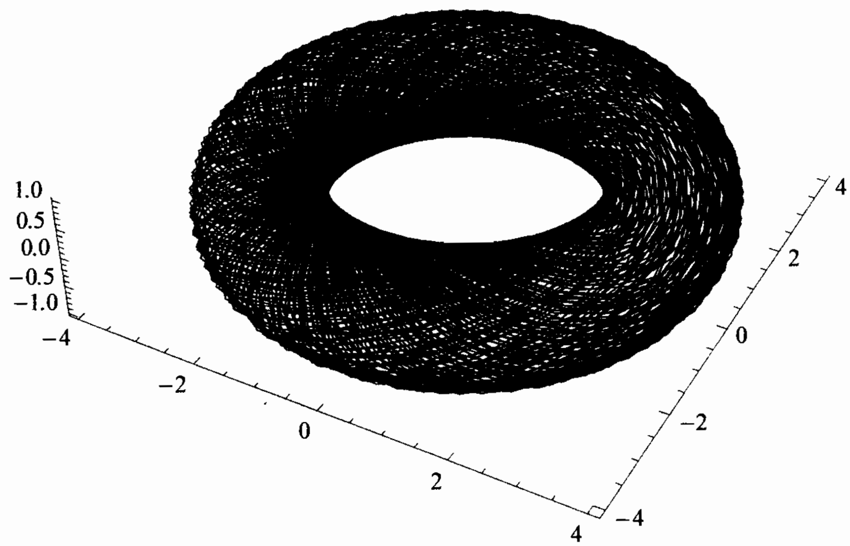
(Pf) $\forall \varepsilon$, $\exists n$ s.t. $\left|\frac{u}{n}\right| < \varepsilon \Rightarrow \exists\, a \in A$ s.t. $\left|a - \frac{u}{n}\right| < \frac{\varepsilon}{n}$
$\Rightarrow |na - u| < \varepsilon$

$-\varepsilon \quad u \quad a \quad +\varepsilon$

(12) again

Homework 2 Problem 12. Mathematical draws densely wound line in torus.

ParametricPlot3D[{(3 + Cos[Sqrt[2] * u]) Cos[u], (3 + Cos[Sqrt[2] * u]) Sin[u], Sin[Sqrt[2] * u]}, {u, 0, 1000}]



p. 240 #13 Define $g(x) = \overset{\text{scalar product}}{\langle x, x \rangle} = \sum_{i=1}^{3} x_i^2 \quad \forall\ x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3$

We claim $g'(x)h = 2\langle x, h\rangle \quad \forall x, h \in \mathbb{R}^3$

Pf

$$\left|\frac{g(x+h) - g(x) - 2\langle x,h\rangle}{\|h\|}\right| = \frac{\langle x+h, x+h\rangle - \langle x,x\rangle - 2\langle x,h\rangle}{\|h\|}$$

$$= \frac{\langle h,h\rangle}{\|h\|} = \frac{\|h\|^2}{\|h\|} = \|h\| \underset{h \to 0}{\longrightarrow} 0$$

QED

Thus, by the chain rule

$h : \mathbb{R} \to \mathbb{R}$

composition of linear maps

$$1 = h(t) = \langle f(t), f(t)\rangle = (g \circ f)(t) \cdot f'(t)$$

$$\Rightarrow 0 = h'(t)h = g'(f(t))(f'(t)h), \quad \forall\ h, t \in \mathbb{R}$$

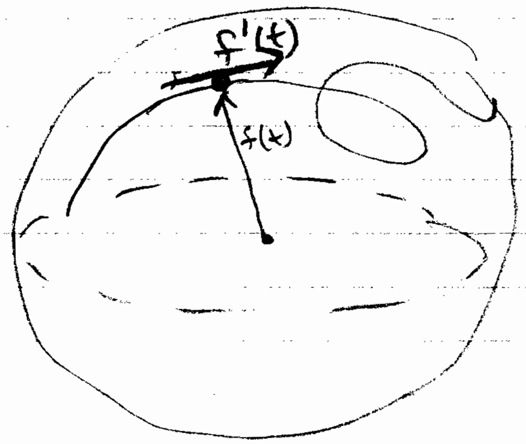
$$0 = 2\langle f(t), f'(t)h\rangle \quad \forall\ h, t \in \mathbb{R}$$

We can take $h = 1$

This says $0 = \langle f(t), f'(t)\rangle$

$\Rightarrow$ $f(t)$ is orthogonal to $f'(t)$

We can view $f'(t)$ as tangent vector to curve $f(t)$ on unit sphere in $\mathbb{R}^3$



p. 240

(14) $f(0,0)=0$, $f(x,y)=\dfrac{x^3}{x^2+y^2}$ for $(x,y)\neq(0,0)$

(a) $D_1 f(x,y) = -\dfrac{2x^4}{(x^2+y^2)^2} + \dfrac{3x^2}{x^2+y^2}$ for $(x,y)\neq(0,0)$

$$D_1 f(0,0) = \lim_{h\to 0}\frac{f(h,0)-0}{h} = \lim_{h\to 0}\frac{\frac{h^3}{h^2}-0}{h} = 1$$

$$D_2 f(x,y) = \frac{-2x^3y}{(x^2+y^2)^2} \quad \text{if } (x,y)\neq(0,0)$$

$$D_2 f(x,y) = \lim_{k\to 0}\frac{f(0,k)-0}{k} = \lim_{k\to 0}\frac{0-0}{k} = 0$$

(d) So f is not differentiable at $(0,0)$ as $D_1 f(0,0)\neq D_2 f(0,0)$

To finish (a) need to bound $D_1 f$ & $D_2 f$.

$(x,y)\neq(0,0) \Rightarrow$

$$|D_1 f| = \left|\frac{(x^2+y^2)(3x^2)-2x^4}{(x^2+y^2)^2}\right| = \frac{x^4+3x^2y^2}{(x^2+y^2)^2} \leq \frac{\frac{3}{2}(x^2+y^2)^2}{(x^2+y^2)^2} = \frac{3}{2}$$

for $(x,y)\neq(0,0)$

$(x,y)\neq(0,0) \Rightarrow$

$$|D_2 f| = \frac{2|x|^3|y|}{(x^2+y^2)^2} = 2\frac{|x|^3\,|y|}{\left\|\binom{x}{y}\right\|^4} = 2\left(\frac{|x|}{\left\|\binom{x}{y}\right\|}\right)^3\left(\frac{|y|}{\left\|\binom{x}{y}\right\|}\right) \leq 2$$

$\dfrac{x}{\left\|\binom{x}{y}\right\|}\leq 1$, as $\left\|\binom{x}{y}\right\| = \sqrt{x^2+y^2} \geq \sqrt{x^2}$ & $\sqrt{y^2}$

So both partials bounded by 2.

Continuity of f at $(x,y)\neq(0,0)$ is clear.

So need to show

$$0 \overset{?}{=} \lim_{(x,y)\to(0,0)}\frac{x^3}{x^2+y^2} = \lim_{(x,y)\to(0,0)} x\left(\frac{x^2}{x^2+y^2}\right)$$

But $0\leq \dfrac{|x^3|}{x^2+y^2} \leq |x| \to 0$ as $(x,y)\to(0,0)$

So by squeeze lemma limit $=0$.

14 b) $\|u\|=1$, $u \in \mathbb{R}^2$, Take $u = \begin{pmatrix} a \\ b \end{pmatrix}$, $a^2+b^2=1$

$$(D_u f)(0,0) = \lim_{t\to 0} \frac{f((0,0)+tu) - f(0,0)}{t}$$

$$= \lim_{t\to 0} \frac{1}{t} \frac{(ta)^3}{t^2(a^2+b^2)} = a^3$$

$|a^3| \le 1$ since $a^2+b^2=1 \Rightarrow a^2 \le 1 \Rightarrow |a| \le 1$

c) $\gamma: \mathbb{R} \to \mathbb{R}^2$ differentiable $\gamma(t) = (x(t), y(t))$

Assume $\|\gamma'(t_0)\| > 0$ if $\gamma(t_0) = (0,0)$

Only problem for $g(t)$ is at such t_0

$g: \mathbb{R} \to \mathbb{R}$. $g(t) = f(x(t), y(t))$.

$$\frac{g(t_0+h) - g(t_0)}{h} = \frac{x(t_0+h)^3}{h\left(x(t_0+h)^2 + y(t_0+h)^2\right)}$$

By γ' differentiable

$$\left.\begin{aligned} x(t_0+h) &= x'(t_0)h + \varphi_1(h) \\ y(t_0+h) &= y'(t_0)h + \varphi_2(h) \end{aligned}\right\} \text{ where}_{i=1,2} \left(\frac{\varphi_i(h)}{|h|} \xrightarrow[h\to 0]{} 0\right)$$

So

$$\frac{g(t_0+h) - g(t_0)}{h} = \frac{1}{h} \frac{\left(x'(t_0)h + \varphi_1(h)\right)^3}{\left(x'(t_0)h + \varphi_1(h)\right)^2 + \left(y'(t_0)h + \varphi_2(h)\right)^2}$$

$$= \frac{1}{h} \frac{x'(t_0)^3 h^3 + 3x'(t_0)^2 h^2 \varphi_1(h) + 3x'(t_0) h \varphi_1^2(h) + \varphi_1^3(h)}{x'(t_0)^2 h^2 + 2x'(t_0) h \varphi_1(h) + \varphi_1^2(h) + y'(t_0)^2 h^2 + 2y'(t_0) h \varphi_2(h) + \varphi_2^2(h)}$$

$$= \frac{x'(t_0)^3 + 3x'(t_0)^2 \frac{\varphi_1(h)}{h} + 3x'(t_0)\frac{\varphi_1^2(h)}{h^2} + \frac{\varphi_1^3(h)}{h^3}}{x'(t_0)^2 + y'(t_0)^2 + \frac{2x'(t_0)\varphi_1(h)}{h} + \frac{\varphi_1^2(h)}{h^2} + \frac{2y'(t_0)\varphi_2(h)}{h} + \frac{\varphi_2^2(h)}{h^2}}$$

$$\xrightarrow[h\to 0]{} \left(\frac{x'(t_0)^3}{x'(t_0)^2 + y'(t_0)^2}\right) = g'(t_0)$$

To see continuity,

we have chain rule if $t \neq t_0$

$$g'(t) = f'(\gamma(t))\,\gamma'(t) \quad \text{, if } t \neq \text{any } t_0 \text{ where } \gamma(t_0)=0$$

$$= \left(\frac{3x^2}{x^2+y^2} - \frac{x^3(2x)}{(x^2+y^2)^2} \;,\; \frac{-x^3 \cdot 2y}{(x^2+y^2)^2} \right) \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$$

$$= \frac{3x^2(x^2+y^2) - 2x^4}{(x^2+y^2)^2}\, x'(t) - \frac{2x^3y}{(x^2+y^2)^2}\, y'(t)$$

Need to see

$$\left| \frac{x'(t_0)^3}{x'(t_0)^2 + y'(t_0)^2} - \frac{(x^4 + 3x^2y^2)x'(t) - 2x^3y \cdot y'(t)}{(x^2+y^2)^2} \right| \underset{t \to t_0}{\longrightarrow} 0$$

As before

$$\begin{cases} x(t_0+h) = x'(t_0)h + \varphi_1(h) \\ y(t_0+h) = y'(t_0)h + \varphi_2(h) \end{cases} , \quad \frac{\varphi_i(h)}{|h|} \to 0 \text{ as } h \to 0$$

Numerator

$$= \begin{cases} (x'(t_0)h + \varphi_1(h))^4 + 3\,(x'(t_0)h + \varphi_1(h))^2 (y'(t_0)h + \varphi_2(h))^2 x'(t) \\ -2\,(x_0'(t_0)h + \varphi_1(h))^3 (y'(t_0)h + \varphi_2(h))\, y'(t) \end{cases}$$

$$= \begin{cases} x'(t_0)^4 h^4 + 4x'(t_0)^3 h^3 \varphi_1 + 6x'(t_0)^2 h^2 \varphi_1^2 + 4x'(t_0) h \varphi_1^3 + \varphi_1^4 \\ +3\,(x'(t_0)^2h^2 + 2x'(t_0)h\varphi_1(h) + \varphi_1^2)(y'(t_0)^2h^2 + 2y'(t_0)\varphi_2 h + \varphi_2^2)\, x'(t) \\ -2\,(x_0'(t_0)^3h^3 + 3x_0'(t_0)^2h^2\varphi_1 + 3x_0'(t_0)h\varphi_1^2 + \varphi_1^3)(y'(t_0)h + \varphi_2)\, y'(t) \end{cases}$$

$$= h^4 \left\{ x'(t_0)^4 + 3x'(t_0)^2 y'(t_0)^2 x'(t) - 2x_0'(t_0)^3 y'(t_0) y'(t) \right\}$$

+ higher order terms in h $o(h^4)$

(i.e. terms that $\to 0$ when divided by h^4)

Denominator will give $h^4 (x'(t_0)^2 + y'(t_0)^2)^2 + o(h^4)$

$$\Longrightarrow \left(\text{as } t = t_0 + h \text{ approaches } t_0 \right) \quad \frac{(x^4 + 3x^2y^2)\,x'(t) - 2x^3y\,y'(t)}{(x^2+y^2)^2(t)} \longrightarrow \star$$

$$\star = \frac{x'(t_0)^4 + 3x'(t_0)^2 y'(t_0)^2 x'(t_0) - 2x_0'(t_0)^3 (y'(t_0))^2}{(x'(t_0)^2 + y_0'(t_0)^2)^2}$$

Let $x'(t_0) \neq 0$

$$= \frac{x'(t_0)^3 \left\{ x'(t_0)^1 + y'(t_0)^2 \right\}}{\left(x'(t_0)^2 + y'(t_0)^2 \right)^2}$$

$$= \frac{\cancel{x'(t_0)^4} \left\{ 1 + \frac{y'(t_0)^2}{x'(t_0)} \right\}}{\cancel{x'(t_0)^4} \left\{ 1 + y'(t_0)^2 / x'(t_0)^2 \right\}^2}$$

Not what we needed:

Here's the difference

which was

$$\frac{x'(t_0)^3}{x'(t_0)^2 + y'(t_0)^2}^2 = \frac{x'(t_0)^3 \left(x_0'(t_0)^2 + y'(t_0)^2 \right)}{\left(x'(t_0)^2 + y'(t_0)^2 \right)^2}$$

only slightly different though

If $y'(t_0) = 0$, then we have 2 different numbers here unless $x'(t_0) = 1$

Rudin seems wrong about continuity of g', unless my algebra had a flaw,

p. 242 #27 $f(0,0)=0$, $f(x,y)=\frac{xy(x^2-y^2)}{x^2+y^2}$ for $(x,y)\neq(0,0)$

a) $$D_1 f = -\frac{2x^2y(x^2-y^2)}{(x^2+y^2)^2}+\frac{2x^2y}{x^2+y^2}+\frac{y(x^2-y^2)}{x^2+y^2}$$

$$D_2 f = -\frac{2xy^2(x^2-y^2)}{(x^2+y^2)^2}-\frac{2xy^2}{x^2+y^2}+\frac{x(x^2-y^2)}{x^2+y^2}$$

Only Need to show $D_i f$ cont. at $(0,0)$. Then $\exists f'(0,0)$ & f cont. at $(0,0)$ by our thm. Let $v=\binom{x}{y}$ $\|v\|^2=x^2+y^2$

$$|D_1 f(x,y)| \le \frac{2|x|^2|y|\ \|v\|^2}{\|v\|^4}+\frac{2|x|^2|y|}{\|v\|^2}+\frac{|y|\ \cancel{\|v\|^2}}{\cancel{\|v\|^2}} \le \left(4\frac{|x|^2|y|}{\|v\|^2}+|y|\right)\le 5|y|$$

$|x|^2\le\|v\|^2=|x|^2+|y|^2$

let $y\to 0$ $\Rightarrow$ $D_1 f(x,y)\to 0$ as $(x,y)\to(0,0)$

$$|(D_2 f)(x,y)| \le \frac{2|x||y|^2(x^2+y^2)}{\|v\|^4}+\frac{2|x||y|^2}{\|v\|^2}+\frac{|x|\ \cancel{\|v\|^2}}{\cancel{\|v\|^2}} \le \left(\frac{4|x||y|^2}{\|v\|^2}+|x|\right)\le 5|x|$$

So $|D_2 f(x,y)|\to 0$ as $(x,y)\to(0,0)$

Thus $D_1 f$ & $D_2 f$ are continuous.

b) $$D_1 f = \frac{-2\{x^4y-x^2y^3\}}{(x^2+y^2)^2}+\frac{3x^2y-y^3}{x^2+y^2} \quad \text{for } (x,y)\neq(0,0)$$

Need $D_2D_1f(0,0)$

$$\lim_{k\to 0}\frac{(D_1 f)(0,k)-\overbrace{D_1 f(0,0)}^{=0}}{k}=\lim_{k\to 0}\frac{1}{k}\left\{\frac{-k^3}{k^2}\right\}=-1=D_2D_1f(0,0)$$

For $(x,y)\neq(0,0)$ we get

$$D_2D_1 f(x,y)=\frac{+4\{x^4y-x^2y^3\}2y}{(x^2+y^2)^3}-\frac{2\{x^4-3x^2y^2\}}{(x^2+y^2)^2}-\frac{(3x^2y-y^3)(2y)}{(x^2+y^2)^2}+\frac{3x^2-3y^2}{x^2+y^2}$$

cont. for $(x,y)\neq(0,0)$

p. 242 # 27 contd

$$D_2 f = \frac{-2(x^3y^2 - xy^4)}{(x^2+y^2)^2} - \frac{3xy^2 - x^3}{x^2+y^2}$$

$$(D_1 D_2 f)(0,0) = \lim_{h\to 0} \frac{(D_2 f)(h,0) - 0}{h} = \lim_{h\to 0} \frac{1}{h}\frac{h^3}{h^2} = +1 = D_1 D_2 f(0,0)$$

For $(x,y) \neq (0,0)$

$$(D_1 D_2 f)(x,y) = \frac{4(2x)(x^3y^2 - xy^4)}{(x^2+y^2)^3} - \frac{2\{3x^2y^2 - y^4\}}{(x^2+y^2)^2} + \frac{3xy^2+x^3}{(x^2+y^2)^2} - \frac{3y^2+3x^2}{x^2+y^2}$$

cont. for $(x,y) \neq (0,0)$

p. 244 #31 $f: \mathbb{R}^2 \to \mathbb{R}$. f''' cont at $a \in U$

$$f'(a) = 0$$

$$f(a+h) = f(a) + f'(a)\cdot h + \frac{f''(a)}{2!} h^{(2)} + R_2$$

where $R_2 = \dfrac{f'''(c)}{3!} h^{(3)}$. ($c$ on line connecting a and $a+h$)

$f'(a) = 0$. So we have

$$f(a+h) = f(a) + \left(\sum_{i,j=1}^{2} \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(a)\right) h_i h_j\right) + R_2$$

Take h so small that $\left|\frac{f''(a)}{2} h^{(2)}\right| \geq 2|R_2|$

We assume this ≥ 0 $\forall\ h \neq 0$.
This says $f(a+h) \geq f(a) + \frac{1}{4} f''(a) h^{(2)}$
Then $f(a+h) \geq f(a)$ for h small.

This condition is that the matrix

$$\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(a)\right)_{1\leq i,j\leq 2}$$ is positive definite.

Equivalently $\boxed{\dfrac{\partial^2 f}{\partial x_1^2}(a) > 0 \text{ and } \det\left(\dfrac{\partial^2 f}{\partial x_i \partial x_j}(a)\right) > 0}$

$$0 < \frac{\partial^2 f}{\partial x_1^2}(a)\frac{\partial^2 f}{\partial x_2^2}(a) - \left(\frac{\partial f}{\partial x_1 \partial x_2}(a)\right)^2$$

Problem A $\quad f: U \subset \mathbb{R}^n \to \mathbb{R}^m$ diffbl at $c \in U \overset{?}{\iff} \forall j,\ f_j$ diffbl at c

where $f(x) = \begin{pmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{pmatrix}$

Pf. $(\Rightarrow)$ $\exists f'(c)$

$$f(c+h) - f(c) = \begin{pmatrix} f_1(c+h) - f_1(c) \\ \vdots \\ f_m(c+h) - f_m(c) \end{pmatrix} = f'(c) \cdot h + \|h\| \psi(h)$$

$\psi(h) \to 0$ as $h \to 0$

$f'(c)$ is an $m \times n$ matrix, $h \in \mathbb{R}^n$, $h = \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}$

$$\psi(h) = \begin{pmatrix} \psi_1(h) \\ \vdots \\ \psi_m(h) \end{pmatrix} \in \mathbb{R}^m$$

So

$$f(c+h) - f(c) = \begin{pmatrix} f_1(c+h) - f_1(c) \\ \vdots \\ f_m(c+h) - f_m(c) \end{pmatrix} = \begin{pmatrix} f'_{11}(c) & \cdots & f'_{1n}(c) \\ \vdots & & \\ f'_{m1}(c) & \cdots & f'_{mn}(c) \end{pmatrix} \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} + \|h\| \begin{pmatrix} \psi_1(h) \\ \vdots \\ \psi_m(h) \end{pmatrix}$$

$\iff \forall j$

$$f_j(c+h) - f_j(c) = \begin{pmatrix} f'_{j1}(c) & \cdots & f'_{jn}(c) \end{pmatrix} \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} + \|h\| \psi_j(h)$$

$\iff \exists\ f_j'(c)$