

Math 140c – Homework #3 – due Mon. April 27

(or in drop box by noon of the next day)

- 1) Consider the function $f(0)=0$ and $f(x)=x+x^2\sin(1/x)$, for $x\neq 0$. Show that $f'(0)=1$ but no matter how small we choose $\delta>0$, f is not 1-1 on any interval $(-\delta,\delta)$. Why doesn't this contradict the inverse function theorem?
- 2) Define $f_1(x,y)=y$, $f_2(x,y)=x+y^2$ and $f=(f_1,f_2)$ mapping $\mathbb{R}^2\rightarrow\mathbb{R}^2$.
 - a) Compute the matrix of the derivative $f'(x,y)$ (the Jacobian) and its rank. What does the inverse function theorem say about f ?
 - b) Compute the inverse function g of f and its derivative.
- 3) Suppose U is an open set in \mathbb{R}^n $f:U\rightarrow\mathbb{R}^m$. Suppose that f is continuously differentiable on U . Show that for every a in U and every $\varepsilon>0$ there is a δ (depending on ε and a) such that the ball $B_\delta(a)$ of radius δ about a is contained in U and
$$\|f'(a)h-f'(x)h\|\leq\varepsilon\|h\|$$
 for all h in \mathbb{R}^n and all x in $B_\delta(a)$ and
$$\|f(x)-f(y)-f'(a)(x-y)\|\leq\varepsilon\|x-y\|$$
 for all x,y in $B_\delta(a)$.

Hints. For x,y in $B_\delta(a)$, the line segment joining them lies in $B_\delta(a)$. We can use the mean value theorem to write $(f(x)-f(y))\cdot v=f'(c)(x-y)\cdot v$. Choose $v=f(x)-f(y)-f'(a)(x-y)$. Apply the Cauchy-Schwarz inequality to $v\cdot v$.
- 4) Let U be an open set in \mathbb{R}^n and $f:U\rightarrow\mathbb{R}^m$. Suppose that f is continuously differentiable on U and that for some a in U , $f'(a)$ is 1-1 on \mathbb{R}^n . Show that $\exists\delta>0$ such that the ball $B_\delta(a)$ of radius δ about a is contained in U , $f'(x)$ is 1-1 on $B_\delta(a)$, and the inverse function f^{-1} exists on $f(B_\delta(a))$.

Hints. 1st show that $\exists\mu>0$ s.t. $\|f'(a)h\|\geq 2\mu\|h\|\ \forall h\in\mathbb{R}^n$ since $f'(a)$ is 1-1.

2nd use the continuity of f' on U to show that $\exists\delta>0$ such that $\|f'(x)h\|\geq\mu\|h\|$ for all x in $B_\delta(a)$. This makes $f'(x)$ 1-1 for all x in $B_\delta(a)$.

3rd use problem 3 (taking $\varepsilon=\mu$) to find δ_1 such that $\|f(x)-f(y)\|\geq\mu\|x-y\|$ for any $x,y\in B_{\delta_1}(a)$.
- 5) Apply problem 4) to see that the parameterization map for the torus given in Rudin problem 12 actually provides a topological identification (homeomorphism) of the torus and the square with parallel sides identified. Why is this inverse function continuous?
- 6) $f_1(u,v)=\cos(u)\sin v$ $f_2(u,v)=\sin(u)\cos(v)$ $f_3(u,v)=\cos(v)$. Find $f'(x)$. For what values of u,v do we get a topological identification of a rectangle or square in the plane with the surface of a sphere of radius 1 in 3-space? Explain your answer using the preceding problems.
- 7) Next consider the change from rectangular coordinates in 3-space to spherical polar coordinates in 3-space by introducing the radius into the functions in the preceding problem. Compute the Jacobian matrix and use the inverse function theorem to say where this mapping is invertible.
- 8) Suppose U is an open set in \mathbb{R}^n , $f:U\rightarrow\mathbb{R}^m$. Suppose that f is continuously differentiable on U . Suppose that $f'(x)$ is invertible for all x in U . Show that $n=m$ and $f(U)$ is open in \mathbb{R}^n .