

1) Define and give an example:

- a) Derivative of a function mapping an open set $U \subset \mathbb{R}^n$ into \mathbb{R}^m .
- b) Operator norm of a matrix.
- c) linear map from a vector space into another vector space.

2) Suppose that f is a continuous map from the closed interval $[a,b]$ into \mathbb{R}^m . And suppose that f is differentiable on the open interval (a,b) . Show that there exists a point $c \in (a,b)$ such that

$$\|f(b) - f(a)\| \leq (b-a) \|f'(c)\|.$$

Hint. Let $v = f(b) - f(a)$ and look at the function $g(t) = v \cdot f(t)$, $t \in [a,b]$. Here $v \cdot w$ denotes the scalar or dot product in \mathbb{R}^m . Apply the one-variable mean value theorem to $g(t)$ and use the Cauchy-Schwarz inequality.

3) Suppose f maps a convex open set $U \subset \mathbb{R}^n$ into \mathbb{R}^m . Assume that f is differentiable.

- a) Prove that if $f'(x) = 0$ for all x in U , then f is constant on U .
- b) Suppose $\|f'(x)\| < \frac{1}{2}$ for all $x \in U$. Let X be a closed subset of U . Show that then f has a unique fixed point in X .

4) Consider the function $f(x,y) = xy^2 / (x^2 + y^2)$ for $(x,y) \neq (0,0)$ and $f(0,0) = 0$. Is f differentiable at $(0,0)$?

Hint. Find the 1st partials at $(0,0)$. Then look at $g(t) = f(tu, tv)$ for a unit vector (u,v) . Find $g'(0)$ in 2 ways.

5) a) Look at the function $f(x,y) = (x^2 + y^2) \sin(1/(x^2 + y^2))$ for $(x,y) \neq (0,0)$ and $f(0,0) = 0$. Show that the 1st partial derivative exist everywhere and they are not continuous at $(0,0)$.

b) Is $f(x,y)$ differentiable at $(0,0)$? Why?

Hint on a). Look at the sequence in the plane: $(1/n, 0)$, $n = 1, 2, 3, \dots$

6) **True-False.**

Tell whether the following statements are true or false. If true, give a brief reason for your answer. If false give a counterexample.

- a) Suppose $L: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is linear. Then $L' = L$.
- b) Suppose f maps an open set $U \subset \mathbb{R}^2$ into \mathbb{R} . If the mixed 2nd partials of f exist and are both continuous at some point c in U , then they must be equal at c .
- c) Suppose f maps an open set $U \subset \mathbb{R}^n$ into \mathbb{R}^m . If both first partials exist at (a,b) in U , then f is differentiable at (a,b) .
- d) Suppose that f is a continuous map from the closed interval $[a,b]$ into \mathbb{R}^2 . Suppose that f is differentiable on (a,b) . Then there is a point c in (a,b) such that

$$f(b) - f(a) = f'(c)(b-a).$$

- 7) Suppose that f, g maps the open set $U \subset \mathbb{R}^n$ into \mathbb{R}^n . Define $h(x) = f(x) \bullet g(x)$ to be the scalar or dot product of $f(x)$ and $g(x)$. Using the definition of derivative, find the derivative $h'(x)$, assuming both f and g differentiable at x .
- 8) Suppose f maps an open set $U \subset \mathbb{R}^2$ into \mathbb{R} . Assume that $f^{(3)}(x)$ continuous on U .
- State the first and second derivative tests for (a, b) to be a minimum of $f(x, y)$. Here you need to know what a positive definite symmetric matrix M is. ($\forall v Mv \succ 0$ for all non-0 vectors v).
 - Prove them using Taylor's formula.
 - Apply them to find the maxima and minima of $f(x, y) = (x + y)e^{-x^2 - y^2}$.
- 9) Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be given by $f(x, y) = (e^x \cos y, e^x \sin y)$. Show that $f'(x, y)$ is invertible for all $(x, y) \in \mathbb{R}^2$. Deduce that f is locally invertible at every point. Show that $f(x, y)$ does not have a global inverse on all of \mathbb{R}^2 .
- 10) Suppose $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ is a continuously differentiable function and that $D_2 f(a, b) \neq 0$. Let g be a function given by the implicit function theorem that solves the equation $f(x, g(x)) = 0$ with $g(a) = b$. Show that

$$g'(x) = \frac{-D_1 f(x, g(x))}{D_2 f(x, g(x))}.$$

