

1) Define and give an example:

- a) Derivative of a function mapping an open set  $U \subset \mathbb{R}^n$  into  $\mathbb{R}^m$ .
- b) Operator norm of a matrix.
- c) linear map from a vector space into another vector space.

2) Suppose that  $f$  is a continuous map from the closed interval  $[a,b]$  into  $\mathbb{R}^m$ . And suppose that  $f$  is differentiable on the open interval  $(a,b)$ . Show that there exists a point  $c \in (a,b)$  such that

$$\|f(b) - f(a)\| \leq (b-a)\|f'(c)\|.$$

**Hint.** Let  $v=f(b)-f(a)$  and look at the function  $g(t)=v \bullet f(t)$ ,  $t \in [a,b]$ . Here  $v \bullet$  denotes the scalar or dot product in  $\mathbb{R}^m$ . Apply the one-variable mean value theorem to  $g(t)$  and use the Cauchy-Schwarz inequality.

3) Suppose  $f$  maps a convex open set  $U \subset \mathbb{R}^n$  into  $\mathbb{R}^m$ . Assume that  $f$  is differentiable.

- a) Prove that if  $f'(x)=0$  for all  $x$  in  $U$ , then  $f$  is constant on  $U$ .
- b) Suppose  $\|f'(x)\| < \frac{1}{2}$  for all  $x \in U$ . Let  $X$  be a closed subset of  $U$ . Show that then  $f$  has a unique fixed point in  $X$ .

4) Consider the function  $f(x,y)=xy^2/(x^2+y^2)$  for  $(x,y) \neq (0,0)$  and  $f(0,0)=0$ . Is  $f$  is differentiable at  $(0,0)$ ?

**Hint.** Find the 1<sup>st</sup> partials at  $(0,0)$ . Then look at  $g(t)=f(tu,tv)$  for a unit vector  $(u,v)$ . Find  $g'(0)$  in 2 ways.

5) a) Look at the function  $f(x,y)=(x^2+y^2)\sin(1/(x^2+y^2))$  for  $(x,y) \neq (0,0)$  and  $f(0,0)=0$ . Show that the 1<sup>st</sup> partial derivative exist everywhere and they are not continuous at  $(0,0)$ .

b) Is  $f(x,y)$  differentiable at  $(0,0)$ ? Why?

**Hint on a).** Look at the sequence in the plane:  $(1/n,0)$ ,  $n=1,2,3, \dots$

6) **True-False.**

**Tell whether the following statements are true or false. If true, give a brief reason for your answer. If false give a counterexample.**

- a) Suppose  $L: \mathbb{R}^n \rightarrow \mathbb{R}^m$  is linear. Then  $L^t=L$ .
- b) Suppose  $f$  maps an open set  $U \subset \mathbb{R}^2$  into  $\mathbb{R}$ . If the mixed 2<sup>nd</sup> partials of  $f$  exist and are both continuous at some point  $c$  in  $U$ , then they must be equal at  $c$ .
- c) Suppose  $f$  maps an open set  $U \subset \mathbb{R}^n$  into  $\mathbb{R}^m$ . If both first partials exist at  $(a,b)$  in  $U$ , then  $f$  is differentiable at  $(a,b)$ .
- d) Suppose that  $f$  is a continuous map from the closed interval  $[a,b]$  into  $\mathbb{R}^2$ . Suppose that  $f$  is differentiable on  $(a,b)$ . Then there is a point  $c$  in  $(a,b)$  such that
 
$$f(b)-f(a)=f'(c)(b-a).$$

- 7) Suppose that  $f, g$  maps the open set  $U \subset \mathbb{R}^n$  into  $\mathbb{R}^n$ . Define  $h(x) = f(x) \bullet g(x)$  to be the scalar or dot product of  $f(x)$  and  $g(x)$ . Using the definition of derivative, find the derivative  $h'(x)$ , assuming both  $f$  and  $g$  differentiable at  $x$ .
- 8) Suppose  $f$  maps an open set  $U \subset \mathbb{R}^2$  into  $\mathbb{R}$ . Assume that  $f^{(3)}(x)$  continuous on  $U$ .
- State the first and second derivative tests for  $(a, b)$  to be a minimum of  $f(x, y)$ . Here you need to know what a positive definite symmetric matrix  $M$  is. ( $\forall v Mv > 0$  for all non-0 vectors  $v$ ).
  - Prove them using Taylor's formula.
  - Apply them to find the maxima and minima of  $f(x, y) = (x + y)e^{-x^2 - y^2}$ .
- 9) Let  $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be given by  $f(x, y) = (e^x \cos y, e^x \sin y)$ . Show that  $f'(x, y)$  is invertible for all  $(x, y) \in \mathbb{R}^2$ . Deduce that  $f$  is locally invertible at every point. Show that  $f(x, y)$  does not have a global inverse on all of  $\mathbb{R}^2$ .
- 10) Suppose  $f: \mathbb{R}^2 \rightarrow \mathbb{R}$  is a continuously differentiable function and that  $D_2 f(a, b) \neq 0$ . Let  $g$  be a function given by the implicit function theorem that solves the equation  $f(x, g(x)) = 0$  with  $g(a) = b$ . Show that

$$g'(x) = \frac{-D_1 f(x, g(x))}{D_2 f(x, g(x))}.$$

