# Practice Exam 1 Solutions

1) a) If $\exists$ a linear map $L: \mathbb{R}^n \to \mathbb{R}^m$ and a function $\varphi(h)$ defined for $\|h\|$ small enough s.t. $\lim_{h\to 0} \frac{\varphi(h)}{\|h\|} = 0$ and

$$f(x+h) = f(x) + L(h) + \varphi(h),$$

then we say $L = f'(x)$.

<u>Example</u>: If $f$ is itself linear, then $f' = f$

b) $\|M\| = \max_{\|x\|=1} \{ Mx \mid x \in \mathbb{R}^n \}$ if $M \in \mathbb{R}^{m \times n}$

<u>Example</u>: $M = I$ = identity $\Rightarrow \|I\| = 1$

c) $L: V \to W$ linear ($V, W$ vector spaces/ $\mathbb{R}$)

$\Leftrightarrow L(\alpha x + \beta y) = \alpha Lx + \beta Ly \quad \forall\, x, y \in V, \forall\, \alpha, \beta \in \mathbb{R}$,

<u>Example</u>. $V = W = \mathbb{R}^n$, $Lv = v$ = identity

or let $V = \mathbb{R}^n$, $W = \mathbb{R}$ & $a \in \mathbb{R}^n$. Let $L(v) = a \cdot v \;\forall\, v \in \mathbb{R}$.

2) $v = f(b) - f(a)$, $g(t) = v \cdot f(t)$, dot product.

Apply the 1-variable mean value thm to $g(t)$.

Product rule says $\underline{g'(t) = 0 + v \cdot f'(t)}$ (as $\frac{dv}{dt} = 0$)

(See Rudin p. 113)

So we see

$$g(b) - g(a) = (v \cdot f'(c))(b-a), \text{ for some } c \in (a, b)$$

$$\Rightarrow v \cdot (f(b) - f(a)) = v \cdot f'(c)\,(b-a)$$

$$\Rightarrow v \cdot v = v \cdot f'(c)\,(b-a)$$

Cauchy-Schwarz $\Rightarrow \|v\|^2 \le \|v\| \, \|f'(c)\| \, (b-a)$

$v = f(b) - f(a)$, $\|v\| \neq 0$ WLOG $\Rightarrow \|f(b) - f(a)\| \le \|f'(c)\|\,(b-a)$

3) a) Use the mean value inequality to see $\|f'(x)\| \le M$ on $U$ $\Rightarrow \|f(b) - f(a)\| \le M \|b - a\| \;\forall\, a, b \in U$

We can take $M = 0$ so $f(b) = f(a)$

(Mean value $\le$ proved by letting $\gamma(t) = (1-t)a + tb$ for $t \in [0,1]$ & using #2. & fact that $g(t) = f(\gamma(t))$ satisfies hypotheses of 2) )

$\boxed{\text{Assume } f(X) \subset X}$

3b) $\|f'(x)\| < \frac{1}{2} \quad \forall\ x \in U \implies \|f(b) - f(a)\| \le \frac{1}{2}\|b-a\|$
$\forall\ a, b \in U$ by mean-value $\le$.
Thus $f(x)$ is a shrinking fn on $X$ complete,
So fixed pt $= \lim_{n\to\infty} {}^n f(x)$ for any $x \in X$.
Here ${}^n f = \underbrace{f \circ \cdots \circ f}_{n}$. To see $x_n = {}^n f(x)$ is Cauchy
note

$$\|x_{n+1} - x_n\| \le \frac{1}{2}\|x_n - x_{n-1}\| \le \left(\tfrac{1}{2}\right)^2 \|x_{n-1} - x_{n-2}\| \le \cdots \le \left(\tfrac{1}{2}\right)^{n-1}\|x_2 - x_1\| \le \tfrac{1}{2}^n \|x_1 - x\|$$

$$n > m \implies \|x_n - x_m\| \le \|x_n - x_{n-1}\| + \|x_{n-1} - x_{n-2}\| + \cdots + \|x_{m+1} - x_m\|$$
$$\le \left(\tfrac{1}{2}^{n-1} + \cdots + \tfrac{1}{2}^m\right)\|x_1 - x\|$$
$$\le \tfrac{1}{2}^m \frac{1}{1-c}\|x_1 - x\| \to 0 \text{ as } m \to \infty$$

$\implies \{x_n\}$ Cauchy & $X$ complete $\implies$ $x_n$ has limit $a$

$$a = \lim_{n\to\infty} {}^n f(x) \implies f(a) = f \lim_{n\to\infty} \underbrace{f \circ \cdots \circ f}_{n}(x) = \lim_{n\to\infty} \underbrace{f \circ f \circ \cdots \circ f}_{n+1}(x) = a$$

4) $f(x, y)$'s differentiability is clear except at $(0,0)$
Let's try to show $f'(0,0)h = 0 \quad \forall\ h \in \mathbb{R}^2$

$$D_1 f(0,0) = \lim_{h\to 0} \frac{|f(h,0) - f(0,0)|}{h} = 0, \quad D_2 f(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = 0$$

So if $f'(0,0)$ exists it must have matrix $(0,0)$
For unit vector $u = (u_1, u_2)$ look at $f(tu_1, tu_2) = g(t)$
If $f$ were differentiable at $(0,0)$, $g'(0) = f'(0,0)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0$

$$g'(0) = \lim_{t\to 0} \frac{f(tu_1, tu_2) - f(0,0)}{t} = \lim_{t\to 0} \frac{t^3 u_1 u_2^3}{t^3(u_1^2 + u_2^2)} = u_1 u_2^3 \ne 0$$

$u_1^2 + u_2^2 = 1$

if $u_1$ & $u_2$ both are $\ne 0$

So $f$ can't have a derivative at $0$.

5) We just need to consider $(0,0)$
$(D_1 f)(0,0) = \lim_{h\to 0} h^2 \sin\frac{1}{h^2} \le \lim_{h\to 0} h^2 = 0$ similarly $D_2 f(0,0) = 0$
For $(x,y) \ne (0,0)$

$$(D_1 f)(x,y) = 2x \sin\left(\frac{1}{x^2+y^2}\right) - (x^2+y^2)\cos\left(\frac{1}{x^2+y^2}\right)\frac{(2x)}{(x^2+y^2)^2}$$

$$= 2x \sin\frac{1}{x^2+y^2} - \frac{2x}{x^2+y^2}\cos\left(\frac{1}{x^2+y^2}\right)$$

This not continuous at $(0,0)$. Look at sequence
$\left(\frac{1}{k}, 0\right) \quad (D_1 f)\left(\frac{1}{k}, 0\right) = 2k \sin(k^2) - \frac{2}{k} \cdot \frac{k^2}{4} \cos(k^2)$
$= 2k \sin k^2 - \frac{k}{2}\cos(k^2) \not\to$ limit as $k \to \infty$

5 (contd)

$$\frac{f(h,k)-f(0,0)}{\sqrt{h^2+k^2}} = \sqrt{h^2+k^2}\,\sin\frac{1}{h^2+k^2} \to 0 \quad \text{as } (h,k)\to(0,0)$$

$$\Rightarrow \exists\, f'(0,0) = (0,0)$$

6 a) T (at least if we mean $L'(v) = L$)

$$\frac{L(v+h) - Lv - Lh}{\|h\|} = 0 \Rightarrow L'(v)h = Lh$$

$$\Rightarrow L'(v) = L \quad \forall v \in \mathbb{R}^n$$

b) T It is a theorem. We don't even have to assume anything about $D_2 f$

See Rudin, p. 235

c) F See #4 for a counter example

d) F $f(t) = \begin{pmatrix} t^2 \\ t^3 \end{pmatrix}$ on $[0,2]$

$$f(2) - f(0) = \begin{pmatrix} 4 \\ 8 \end{pmatrix} = f'(c)\,(2)$$

$$\begin{pmatrix} 4 \\ 8 \end{pmatrix} = 2\begin{pmatrix} 2c \\ 3c^2 \end{pmatrix} = \begin{pmatrix} 4c \\ 6c^2 \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} 4 = 4c \\ 8 = 6c^2 \end{cases} \Leftrightarrow \begin{matrix} 1 = c \\ \frac{4}{3} = c^2 \end{matrix}$$

Not possible

7) $\varphi(v,w) = v\cdot w \qquad \varphi: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

Claim: $\varphi'(v,w)\begin{pmatrix}h\\k\end{pmatrix} = v\cdot k + w\cdot h$

$$\frac{|\varphi(v+h, w+k) - \varphi(v,w) - (v\cdot h + w\cdot k)|}{\sqrt{\|h\|^2 + \|k\|^2}}$$

$$= \frac{|(v+h)\cdot(w+k) - v\cdot w - v\cdot k - h\cdot w|}{\sqrt{\|h\|^2+\|k\|^2}}$$

$$= \frac{|h\cdot k|}{\sqrt{\|h\|^2+\|k\|^2}} \leq \frac{\|h\|\cdot\|k\|}{\sqrt{\|h\|^2+\|k\|^2}} \leq \frac{\frac{1}{2}(\|h\|^2+\|k\|^2)}{\sqrt{\|h\|^2+\|k\|^2}} \to 0 \text{ as } h,k\to 0$$

$0 \leq (\|h\| - \|k\|)^2$

$2\|h\|\,\|k\| \leq \|h\|^2 + \|k\|^2$

So let $h(x) = \varphi(f(x), g(x))$

$h'(x)h = \varphi'(f(x), g(x))\,(f'(x)h,\ g'(x)h)$

$$\boxed{h'(x) = f(x)(g'(x)h) + g(x)(f'(x)h)}$$

(8) a) If $f(a,b)$ is a local min (or max) of $f$ for some $(a,b)\in U$, then $f'(a) = 0$.

Proof Look at $f(a+th, b+tk)$, $t$ small real

$f(0)$ is a local max $\Rightarrow f'(0) = 0$

$\Rightarrow f'(a,b)\begin{pmatrix}h\\k\end{pmatrix} = 0 \quad \forall \begin{pmatrix}h\\k\end{pmatrix} \Rightarrow f'(a,b) = (0,0)$

b) Taylor's formula says

$$f(x,y) = f(a,b) + D_1f(a,b)(x-a) + D_2f(a,b)(y-b) + \frac{1}{2}\{D_{11}f(a,b)(x-a)^2 + 2D_{12}f(a,b)(x-a)(y-b) + D_{22}f(a,b)(y-b)^2\} + R_2$$

where since $f^{(3)}(x)$ is continuous we can estimate $R_3$ to be less than the $2^{nd}$ order term for $(x,y)$ near $(a,b)$.

Thus we can say $f(a,b)$ a local max $\Leftrightarrow$

$$(h_1\ h_2)\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}(a,b)\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \geq 0$$

$\forall$ vectors $h = \binom{h_1}{h_2} \in \mathbb{R}^2$.

This says the matrix is positive definite.
$\Leftrightarrow \frac{\partial^2 f}{\partial x^2} > 0$ & $\det > 0$

c) $$\left(\frac{\partial f}{\partial x}\ \frac{\partial f}{\partial y}\right) = \left(e^{-x^2-y^2} + (x+y)(-2x)e^{-x^2-y^2},\ e^{-x^2-y^2} + (x+y)(-2y)e^{-x^2-y^2}\right) = (0,0)$$

$$\Leftrightarrow \begin{cases} 1 - 2x(x+y) = 0 \\ 1 - 2y(x+y) = 0 \end{cases} \Leftrightarrow \begin{cases} 2x(x+y) = 1 \\ 2y(x+y) = 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} x = \frac{1}{2(x+y)} \\ y = \frac{1}{2(x+y)} \end{cases} \Rightarrow \left(x = y \Rightarrow 4x^2 = 1 \Rightarrow x = \pm\frac{1}{2}\right)$$

So we need to look at $\left(\frac{1}{2},\frac{1}{2}\right), \left(-\frac{1}{2},-\frac{1}{2}\right)$

$$\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = e^{-x^2-y^2}\begin{pmatrix} -6x + 4x^3 - 2y + 4x^2y & -2y + 4xy^2 - 2x + 4x^2y \\ -2y + 4x^2y - 2x + 4xy^2 & -6y + 4y^3 - 2x + 4xy^2 \end{pmatrix}$$

at $\left(\frac{1}{2},\frac{1}{2}\right)$

$$f''\left(\tfrac{1}{2},\tfrac{1}{2}\right) = e^{-\frac{1}{2}}\begin{pmatrix} -3 & -1 \\ -1 & -3 \end{pmatrix} = -e^{-\frac{1}{2}}\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \Rightarrow$$ max at $\left(\frac{1}{2},\frac{1}{2}\right)$

$\boxed{f\left(\tfrac{1}{2},\tfrac{1}{2}\right) = e^{-\frac{1}{2}}}$

$$f''\left(-\tfrac{1}{2},-\tfrac{1}{2}\right) = e^{-\frac{1}{2}}\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

$3 > 0$
$\det = 8 > 0$
min at $\left(-\frac{1}{2},-\frac{1}{2}\right)$

$\boxed{f\left(-\tfrac{1}{2},-\tfrac{1}{2}\right) = -e^{-\frac{1}{2}}}$

9) $$f'(x,y) = \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix} = e^x \begin{pmatrix} \cos y & -\sin y \\ \sin y & \cos y \end{pmatrix}$$

$$\Rightarrow \quad \det f'(x,y) = e^x \neq 0$$

Inverse Fn. Thm $\Rightarrow$ $f$ has a local inverse

But $f(x,y) = f(x, y + 2\pi n)$ $\forall\ n \in \mathbb{Z}$

not 1-1

no global inverse

10) use chain rule on $h(x) = f(x, g(x)) = 0$

$$0 = h'(x) = \begin{pmatrix} D_1 f & D_2 f \end{pmatrix}(x, g(x)) \cdot \begin{pmatrix} 1 \\ g'(x) \end{pmatrix}$$

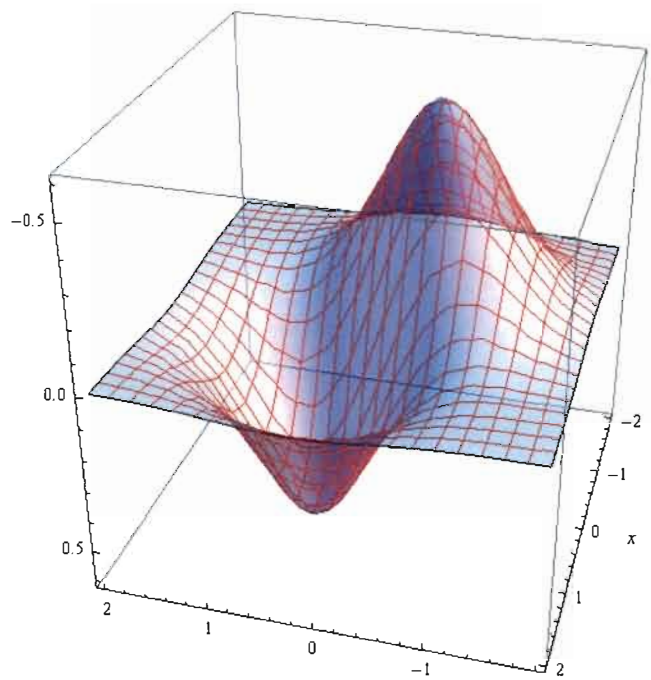
$$\Rightarrow \quad 0 = D_1 f + D_2 f\, g'(x)$$

$$\Rightarrow \quad \boxed{g'(x) = -\frac{D_1 f}{D_2 f}(x, g(x))}$$

## Mathematica does Problem 8c)

In[12]:= Plot3D[(x + y) * Exp[-x^2 - y^2], {x, -2, 2}, {y, -2, 2}, MeshStyle → Red, Mesh → 20, BoxRatios -> {1, 1, 1}, PlotStyle → Opacity[0.5], AxesLabel → Automatic]

Out[12]=




In[22]:= MatrixForm[D[(x + y) * Exp[-x^2 - y^2], {{x, y}, 1}] // Simplify]

Out[22]//MatrixForm=

$$\begin{pmatrix} e^{-x^2-y^2}\left(1-2x^2-2xy\right) \\ e^{-x^2-y^2}\left(1-2xy-2y^2\right) \end{pmatrix}$$

In[20]:= MatrixForm[D[(x + y) * Exp[-x^2 - y^2], {{x, y}, 2}] // Simplify]

Out[20]//MatrixForm=

$$\begin{pmatrix} e^{-x^2-y^2}\left(-6x+4x^3-2y+4x^2y\right) & e^{-x^2-y^2}\left(-2y+4x^2y+x\left(-2+4y^2\right)\right) \\ e^{-x^2-y^2}\left(-2y+4x^2y+x\left(-2+4y^2\right)\right) & e^{-x^2-y^2}\left(-6y+4y^3+x\left(-2+4y^2\right)\right) \end{pmatrix}$$

In[23]:= Solve[{$1 - 2x^2 - 2xy = 0$, $1 - 2xy - 2y^2 = 0$}, {x, y}]

Out[23]= $\left\{\left\{x \to -\frac{1}{2}, y \to -\frac{1}{2}\right\}, \left\{x \to \frac{1}{2}, y \to \frac{1}{2}\right\}\right\}$

A = D[Exp[.5] * (x + y) * Exp[-x^2 - y^2], {{x, y}, 2}] /. {x -> .5, y → .5}

Out[9]= {{-3., -1.}, {-1., -3.}}

In[8]:= B = D[Exp[.5] * (x + y) * Exp[-x^2 - y^2], {{x, y}, 2}] /. {x → -.5, y → -.5}

Out[8]= {{3., 1.}, {1., 3.}}

In[31]:= ContourPlot[(x + y) * Exp[-x^2 - y^2], {x, -2, 2}, {y, -2, 2}, Contours → 50, ColorFunction → Hue]

Out[31]=