



Math 140c, Winter, 2009 Practice Exam 2 - The exam will be much shorter!



- 1) Define and give an example:
- a) Integrable function on a rectangle R in the plane.
 - b) negligible subset $A \subset \mathbb{R}^2$
 - c) admissible function $f: R \rightarrow \mathbb{R}$ for a rectangle R in the plane
 - d) partition P of a rectangle of size $< \delta$
 - e) admissible subset $A \subset \mathbb{R}^2$
 - f) admissible function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$
 - g) admissible function $f: A \rightarrow \mathbb{R}$ for an admissible set $A \subset \mathbb{R}^2$
 - h) set of Lebesgue measure 0 in the plane
 - i) $d\omega$ for a differential form ω
 - j) pull back or inverse image $\sigma^*\omega$ of differential form ω on V under C^∞ map $\sigma: U \rightarrow V$, for U, V open in plane
- 2) Suppose that $g: [a, b] \rightarrow \mathbb{R}$ is continuous. Show that the graph of g (i.e., the set $\{(x, g(x)) \mid x \in [a, b]\}$) is a negligible subset of \mathbb{R}^2 .
- 3) Suppose that we have a continuous function $f: R \rightarrow \mathbb{R}$ for a rectangle R in the plane. Show that f is integrable on R .
- 4) Suppose that now the function f in the last problem has discontinuities at a negligible subset of R but is still bounded on R . Show that f is still integrable on R .
- 5) Suppose that A is an admissible subset of the plane and S is a negligible set such that we have a disjoint union $A = A_1 \cup S \cup A_2$. Show that for an admissible function f on A , we have $\iint_A f = \iint_{A_1} f + \iint_{A_2} f$.
- 6) Show that $\iint_A f$ is linear in f (for A admissible and f integrable).
- Hint.** In order to make your life easier with negative scalars and to eliminate infs and sups from your life, use arbitrary Riemann sums rather than upper and lower ones. Or just note that if the scalar α is negative, then $U(\alpha f, \mathcal{P}) = \alpha L(f, \mathcal{P})$. There is an error on Lang, p. 569 related to this.
- 7) Show that $f \leq g$ on A implies $\iint_A f \leq \iint_A g$ assuming A admissible and f, g integrable.
- 8) Let R be a rectangle in the plane and f an integrable function on R . Suppose that for each rectangle $S \subset R$ we have a number $I(f, S)$ having the following 2 properties.
- Int1.** If \mathcal{P} is a partition of R then $I(f, R) = \sum_S I(f, S)$, where the sum is over the subrectangles of \mathcal{P} .
- Int2.** If we have numbers m, M such that $m \leq f(x) \leq M$, for all $x \in R$, then $m \text{ area}(S) \leq I(f, S) \leq M \text{ area}(S)$.
- Show that $I(f, R) = \iint_R f$.

9) True-False.

Tell whether the following statements are true or false. If true, give a brief reason for your answer. If false give a counterexample.

a) Suppose that we have a function defined on the rectangle $[0,1]^2$ and that the iterated integral

$$\int_0^1 \left(\int_0^1 f(x,y) dy \right) dx \text{ exists. Then the double integral } \iint_{\{0,1\}^2} f(x,y) dx dy \text{ exists.}$$

b) $\left| \iint_A fg \right|^2 \leq \int_A |f|^2 \int_A |g|^2$, assuming that f and g are admissible and on the admissible set A in the plane.

c) Suppose that the admissible functions f and g on the admissible set A are such that $f=g$ except on a negligible subset S of A . Then

$$\iint_A f = \iint_A g .$$

d) Suppose f is a differential form of degree 0; (i.e, f is a function on an open set in the plane). Then $d^2f=0$.

10) Suppose R is a rectangle in the plane and that $\sigma(x,y)=(u,v)$ is continuously differentiable on an open set containing R with a continuously differentiable inverse. Show using Green's theorem that the area of $D=\sigma(R)$ is given by $\iint_D |\det(\sigma')|$.

11) Suppose that S is an admissible region in the plane. Consider the Dirichlet eigenvalue problem for the Laplace operator: $\Delta u = u_{xx} + u_{yy} = -\lambda u$ on the interior S° ; $u=0$ on the boundary ∂S . Show using Green's theorem that we can write the eigenvalue λ as a Rayleigh quotient

$$\lambda = \frac{\iint_S (u_x^2 + u_y^2)}{\iint_S u^2} .$$

Conclude that $\lambda \geq 0$.

12) Suppose that U, V are open sets in the plane, $\sigma: U \rightarrow V$ is C^∞ , and f is a function on V . Show that $\sigma^*df = d(\sigma^*f)$.

13) Suppose A is an $n \times n$ real matrix. Let e_1, \dots, e_n be the standard basis of \mathbb{R}^n . Show that

$$Ae_1 \wedge \dots \wedge Ae_n = \det(A) e_1 \wedge \dots \wedge e_n .$$

14) Consider the polar coordinate map $(x,y) = \sigma(r,\theta) = (r \cos \theta, r \sin \theta)$. Compute the standard form for $\sigma^*(dx)$, $\sigma^*(dy)$, $\sigma^*(dx \wedge dy)$.

