1) Prove the Generalized Mean Value Theorem also known as Cauchy’s Mean Value Theorem.  
This says the following. Assume that f and g are differentiable on (a,b) and continuous on [a,b]. Show there exists a point $c \in (a,b)$ such that  
$$[f(b)-f(a)]g'(c)=[g(b)-g(a)]f'(c).$$  
Hint. As in the proof of the mean value theorem, consider a function to which you can apply Rolle’s theorem which is Lemma 2.2 in Lang, p. 70, 
$$h(x)=[f(b)-f(a)][g(x)-g(a)]-[g(b)-g(a)][f(x)-f(a)].$$

2) Prove l'Hôpital’s rule. This says the following. Assume that f and g are differentiable on an open interval $(a,c)$. Suppose also that $g(x)$ and $g'(x)$ do not vanish in $(a,c)$. Finally assume that both $f(x)$ and $g(x)$ approach 0 as $x$ goes to $c$, with $x<c$. Then l'Hôpital’s rule says: 
$$\lim_{x \to c} \frac{f'(x)}{g'(x)} = k, \quad \text{then} \quad \lim_{x \to c} \frac{f(x)}{g(x)} = k.$$ 

3) a) Define $g(x) = \begin{cases} e^{-\frac{x}{x-1}}, & \text{for } x > 1 \\ 0, & \text{for } x \leq 1 \end{cases}$. Prove that $g(x)$ is continuous at $x=1$.  

b) Show that the function $g(x)$ from problem 1 is differentiable everywhere. You have to consider 3 cases: $x<1$, $x=1$, and $x>1$.  

4) a) Show that the function $g(x)$ from problem 3 has a second derivative $g''(x)$ for all $x$. Again there are 3 cases.  

b) Using mathematical induction, consider the nth derivative $g^{(n)}(x)$ for the function $g(x)$ of problem 1. Show $g^{(n)}(x)$ exists for all $n \in \mathbb{Z}^+$ and all real numbers $x$. Again there are 3 cases.  

Hint: the induction hypothesis should be something like the following: 
$$g^{(i)}(0) = 0, \text{ for } x \leq 1 \text{ and } g^{(i)}(x) = P_i \left( \frac{1}{x-1} \right) e^{-\frac{x}{x-1}}, \text{ for } x > 1,$$
where $P_i(u)$ is a polynomial.  

5) Sketch a graph of the function $g(x)$ from problem 3. What is the Taylor series for $g(x)$ centered at $x=1$? Does it represent $g(x)$?