1) Set \( h(x) = (f(b) - f(a))(g(x) - g(a)) - (g(b) - g(a))(f(x) - f(a)) \)

Then \( h(x) \) is continuous on \([a, b]\), differentiable on \((a, b)\):

\[
h'(x) = (f(b) - f(a)) g'(x) - (g(b) - g(a)) f'(x).
\]

And

\[
h(a) = (f(b) - f(a)) \cdot 0 - (g(b) - g(a)) \cdot 0 = 0,
\]

\[
h(b) = (f(b) - f(a))(g(b) - g(a)) - (g(b) - g(a))(f(b) - f(a))
\]

\[
\Rightarrow h(b) = 0 = h(a).
\]

So \( h(x) \) satisfies all the conditions of the special case of the mean value theorem, we call Rolle's theorem (notes, p. 57).

This means \( \exists c \in (a, b) \) s.t. \( h'(c) = 0 \).

\[
\Rightarrow 0 = h'(c) = (f(b) - f(a)) g'(c) - (g(b) - g(a)) f'(c).
\]

\[
\Rightarrow (f(b) - f(a)) g'(c) = (g(b) - g(a)) f'(c).
\]

2) Extend \( f(x) \) and \( g(x) \) to continuous functions on \([a, c]\) by setting \( f(c) = g(c) = 0 \).

Fix \( x \in (a, c] \). By problem 1, \( \exists d \in (x, c) \) s.t.

\[
(f(c) - f(x)) g'(d) = (g(c) - g(x)) f'(d)
\]

(Here we replace \( a, b, c \) in #1 with \( x, c, d \).

Since \( f(c) = g(c) = 0 \), this says

\[
-\frac{f(x)}{g(x)} g'(d) = -\frac{g(x)}{g'(d)} f'(d)
\]

We can divide by \( g'(d) g(x) \) as \( g'(d) \neq 0 \)

and \( g(x) \neq 0 \) by hypothesis.

That means

\[
\frac{f(x)}{g(x)} = \frac{f'(d)}{g'(d)}
\]

So, we know \( \forall \varepsilon > 0 \exists \delta_\varepsilon \) s.t. \( a \leq c - \delta < \delta \)

\[
|\frac{f(x)}{g(x)} - L| < \varepsilon
\]

This means \( \delta = \min \{ \delta_\varepsilon, c-a \} \) for \( d \in (x, c) \),

\[
|\frac{f(x)}{g(x)} - L| = |\frac{f'(d)}{g'(d)} - L| < \varepsilon.
\]
3) a) We need to prove
\[ \lim_{x \to 1} q(x) = 0. \]

There is no problem from the left as
\[ q(x) = 0 \text{ if } x \leq 1. \]

So we only need to show
\[ \lim_{x \to 1^-} e^{-\frac{1}{x-1}} = 0. \]

For \( x > 1 \),
\[ 0 \leq \frac{1}{x-1} = \frac{1}{e^{\frac{1}{x-1}}} < \frac{1}{x-1} = x-1. \]

Since \( e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!} = 1 + u + \frac{u^2}{2} + \frac{u^3}{6} + \cdots > u. \)

Thus, \( 0 < x-1 < \delta \Rightarrow 0 < \frac{1}{x-1} < \epsilon. \)

This proves \( \lim_{x \to 1} g(x) = 0 = \lim_{x \to 1} g(x) \). So the
\[ 2\text{-sided limit is } 0. \]

b) Case 1 \( x < 1 \)
\[ g'(x) = 0 \]

Case 2 \( x > 1 \)
\[ g'(x) = \frac{d}{dx} e^{-\frac{x-1}{x-1}} = e^{-\frac{x-1}{x-1}} \cdot \frac{d}{dx} \left( \frac{1}{x-1} \right) = e^{-\frac{x-1}{x-1}} \cdot \frac{1}{(x-1)^2} \]

Case 3 \( x = 1 \)
Here we have to use the definition of the derivative, and in fact we see that the derivative from the left is 0. So we must show the right-hand derivative is also 0.
\[ \lim_{h \to 0} \frac{g(1+h) - g(1)}{h} = e^{-\frac{1}{h}} \cdot 0 = \lim_{h \to 0} \frac{1}{h e^{-\frac{1}{h}}} \]

Again from \( e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!} \), \( 0 < e^{\frac{u^2}{2}} \), we see \( h > 0 \Rightarrow 0 < \frac{1}{h e^{\frac{1}{h}}} < \frac{1}{h e^{\frac{1}{2h}}} = 2h \)

So given \( \varepsilon > 0 \), if \( 0 < h < \frac{\varepsilon}{2} \), we have \( 0 < \frac{1}{h e^{\frac{1}{h}}} < \frac{\varepsilon}{2} 2 = \varepsilon \)

Thus the right-hand derivative is 0 also and \( g'(1) = 0 \).

4) a) \( x < 1 \) \( \Rightarrow \) \( g''(x) = 0 \)

\( x > 1 \) \( \Rightarrow \) \( g''(x) = \frac{d}{dx} \left( e^{-\frac{x}{x-1}} \right) = \frac{d}{dx} \left( e^{-\frac{1}{x-1}} \right) \)

Product Rule \( \Rightarrow g''(x) = \frac{d}{dx} e^{-\frac{x}{x-1}} \cdot \frac{1}{(x-1)^2} + e^{-\frac{x}{x-1}} \cdot \frac{d}{dx} \left( \frac{1}{(x-1)^2} \right) \)

\( = e^{-\frac{x}{x-1}} \cdot \frac{1}{(x-1)^2} + e^{-\frac{x}{x-1}} \cdot \frac{-2}{(x-1)^3} \)

\( g''(x) = e^{-\frac{x}{x-1}} \left\{ \frac{1}{(x-1)^2} - \frac{2}{(x-1)^3} \right\} \)

(\( x = 1 \)) Here we must use the definition of second derivative,
\( g''(1) = \lim_{h \to 0} \frac{g'(1+h) - g'(1)}{h} \)

Clearly for \( h < 0 \) and left hand limit is 0.

So we need only show the right hand limit is 0 as well.
\[ \lim_{h \to 0} \frac{g'(1+h) - g'(1)}{h} = \lim_{h \to 0} \frac{e^{-\frac{1}{h}} \cdot \frac{1}{h^2} - 0}{h} = \lim_{h \to 0} \frac{1}{h^3} e^{-\frac{1}{h}} \]
Again we make the same sort of argument
\[ e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!} \Rightarrow 0 < e^u < \frac{u^4}{4!} = \frac{u^4}{24} \]

So,
\[ 0 < \frac{1}{{n^2} e^{\frac{u}{n}}} < \frac{1}{n^2} \cdot \frac{1}{4!} \cdot 4! = \frac{1}{24} \cdot 4! \cdot 24 \cdot h = 24 \cdot h \]

Thus \( 0 < h < \frac{\varepsilon}{24} \Rightarrow 0 < \frac{1}{h} < \varepsilon \) and the right-hand limit is 0, so that \( g''(l) = 0 \).

b) We've already done the cases \( n=1,2 \) of the induction statement.

So assume
\[ \begin{cases} g^{(n)}(x) = 0 \text{ for } x \leq 1 \\ g^{(n)}(x) = P_n \left( \frac{1}{x-1} \right) e^{-1/(x-1)} \end{cases} \]

where \( P_n \) is a polynomial of degree 2n.

Now prove the same when \( n \) is replaced by \( n+1 \). Again we have 3 cases.

Case 1 \( x < 1 \Rightarrow g^{(n)}(x) = 0 \) and \( \Rightarrow g^{(n+1)}(x) = 0 \)
\[ \forall x < 1 \]

Case 2 \( x > 1 \Rightarrow g^{(n)}(x) = P_n \left( \frac{1}{x-1} \right) e^{-1/(x-1)} \)
So use the product and chain rules to see
\[ g^{(n+1)}(x) = P_n' \left( \frac{1}{x-1} \right) \left( -\frac{1}{(x-1)^2} \right) e^{-1/(x-1)} \]
\[ + P_n \left( \frac{1}{x-1} \right) e^{-1/(x-1)} \left( -\frac{1}{(x-1)^2} \right)^2 \]
\[ \Rightarrow P_{n+1}(u) = P_n(u) (-u^2) + P_n(u) u^2 \]
\[ \text{a polynomial in } u. \text{ The degree of } P_{n+2} \text{ is } 2+ \text{ degree } P_n. \]
Case 3 $x=1$ Now we need limits. Again the left hand limit is 0. So we need only look at $x>1$.

$$q^{(n+1)}(1) = \lim_{h \to 0} \frac{g^{(n)}(1+h) - g^{(n)}(1)}{h}$$

The right hand limit is

$$\lim_{h \to 0, h>0} \frac{P_n(\frac{1}{h}) e^{-\frac{1}{h}} - 0}{-h} = \lim_{h \to 0, h>0} \frac{P_n(\frac{1}{h})}{h} = \lim_{h \to 0, h>0} \frac{Q(x)}{e^x} = 0$$

Why? $Q(x) = \sum_{j=0}^{2n+1} c_j x^j = x P(x)$, with $c_i \in \mathbb{R}$

So we only need to prove $\forall j \in \mathbb{Z}^+$

$$\lim_{x \to \infty} \frac{x^j}{e^x} = 0$$

You could do this by l'Hopital's Rule and induction. Or you could do this as before

$$0 < e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} < \frac{x^{n+1}}{(n+1)!}$$

⇒ $0 < \frac{x^j}{e^x} < \frac{x^{j+1}}{x^{j+1}} \cdot (j+1)! = \frac{(j+1)!}{x} \to 0$ as $x \to \infty.$
5) **Taylor series**

\[
\sum_{n=0}^{\infty} \frac{g^{(n)}(1)}{n!} (x-1)^n = 0
\]

This does not represent \( g(x) \) if \( x > 1 \).