A Few Proofs of Properties of Integrals

Audrey Terras

March 7, 2010

1 Proofs of Most of the Rules for Integrals on pages 80-81.

Here we prove most of the rules for integrals on pages 80-81 of the notes using the 2 Axioms for integrals from p. 76 of the notes and the fundamental theorem of calculus, assuming all functions are continuous on \([a, b]\) and \(a < b\). Most of the proofs derive the rules for integrals from the corresponding rules for derivatives since the fundamental theorem says that integrals are inverse operators to derivatives; that is, derivatives erase integrals and vice versa.

1) Linearity.

Suppose that \(f\) and \(g\) are continuous functions on \([a, b]\) and \(\alpha, \beta \in \mathbb{R}\). Then we want to show that for all \(x \in [a, b]\):

\[
\int_a^x (\alpha f + \beta g) = \alpha \int_a^x f + \beta \int_a^x g.
\]

Take derivatives of the function of \(x\) on the left using the fundamental theorem. You get \((\alpha f + \beta g) (x)\). But the derivative of the function of \(x\) on the right is also \((\alpha f + \beta g) (x)\), using the linearity of derivatives.

So now we know that the left hand function of \(x\) has the same derivative as the right hand function of \(x\). But then a corollary of the mean value theorem says that the two functions differ by a constant \(K\); i.e.,

\[
\int_a^x (\alpha f + \beta g) = \alpha \int_a^x f + \beta \int_a^x g + K.
\]

What is \(K\)? Set \(x = a\) and use the fact that \(\int_a^a h = 0\). This says \(K\) must be 0 and the property is proved. That is, if \(F' = G'\), then \((F - G)' = 0\) and by the Important Special Case of the Mean Value theorem at the bottom of page 58 of these notes, then \(F - G = K\) constant and \(F = G + K\).

2) Substitution.

Suppose that \(g'(x)\) is continuous on \([a, b]\) and \(f\) is continuous on \(g(a, b]\). Then we want to prove that:

\[
\int_{g(a)}^{g(x)} f(u) \, du = \int_a^x f(g(t))g'(t) \, dt.
\]

Again we differentiate the left hand function of \(x\), using the fundamental theorem of calculus and the chain rule, obtaining \(f(g(x))g'(x)\). When we differentiate the right hand side as a function of \(x\), we get the same answer using just the fundamental theorem. Again the mean value theorem tells us that the left and right hand side must differ by a constant. Plug in \(x = a\) to see that the constant must be 0.
3) Integral Preserves ≤.

We want to show assuming $a < b$ and $f(x) \leq g(x)$ for all $x \in [a, b]$ that \[
\int_a^b f \leq \int_a^b g.
\]
To do this, look at $h = g - f$.

Then $h$ is ≥ 0 on $[a, b]$ and Axiom 1 for integrals says $\int_a^b h \geq 0(b - a) = 0$. By the linearity property just proved, then

\[
\int_a^b g - \int_a^b f = \int_a^b h \geq 0.
\]
This implies the result.

4) Positivity.

Assume that $f(x) \geq 0$ and $f(x)$ is continuous for all $x \in [a, b]$. Suppose there is a $c \in (a, b)$ such that $f(c) > 0$. We want to prove that $\int_a^b f > 0$. For this, you should draw a picture. See Figure 1.

By the continuity of $f$ at $c$, if we’re given $\varepsilon = f(c)/2$, then there is a $\delta$ so that $|x - c| < \delta$ implies $|f(x) - f(c)| < f(c)/2$. This means that

\[
\frac{-f(c)}{2} < f(x) - f(c) < \frac{f(c)}{2}.
\]

Add $f(c)$ to this and get, for $x \in (c - \delta, c + \delta)$

\[
0 < \frac{f(c)}{2} < f(x) < \frac{3f(c)}{2}.
\]

It follows using Axiom 2 for integrals and the fact that the integral preserves ≤.
Here we have assumed $a < c < b$. If $a = c$ or $c = b$, the result still works. We leave this to you to prove as an extra credit exercise.