1) Define and give an example:
   a) function \( f : S \rightarrow T \); 1-1 function; onto function; b) denumerable set; c) lub; d) \( \lim_{n \to \infty} x_n \);
   e) If \( f : (c,d) \rightarrow \mathbb{R} \) and \( a \in (c,d) \), define the 2-sided limit \( \lim_{x \to a} f(x) \);
      \( x \neq a, x \in (c,d) \)
   f) Cauchy sequence; g) subsequence.

2) a) State the well ordering axiom for \( \mathbb{Z}^+ \).
    b) Define the absolute value \( |x| \) for a real number \( x \) and state its 3 basic properties.
    c) State the completeness axiom for the real numbers.
    d) State the negation of definition of the 2-sided limit \( \lim_{x \to a} f(x) \) for the function \( f : (c,d) \rightarrow \mathbb{R} \)
       with \( a \in (c,d) \). Hint: You need recall how to negate a statement with the quantifiers \( \forall, \exists \).

3) a) Prove that an increasing sequence of real numbers which is bounded above has a limit.
    b) Show that a Cauchy sequence is bounded.
    c) Show that if a sequence \( \{x_n\} \) is Cauchy and it has a convergent subsequence, then \( \{x_n\} \) converges to the same limit as the subsequence.
    d) Prove that the set \( \mathbb{Z}^+ \) does not have an upper bound in \( \mathbb{R} \).
    e) Suppose that \( \lim_{n \to \infty} x_n = L \) and \( \lim_{n \to \infty} y_n = M \). Prove that \( \lim_{n \to \infty} x_n y_n = LM \).

4) a) Prove that \( \sqrt{5} \) is irrational.
    b) Show that \( \mathbb{Q} \) is denumerable.
    c) Show that \( \mathbb{R} \) is not denumerable.
    d) State and prove the formula for the number of functions \( f : S \rightarrow T \), if \( S \) and \( T \) are finite sets.
    e) Prove by induction that for each positive integer \( n \) and for any real \( x \geq -1 \), we have
       \( (1+x)^n \geq 1 + nx \).
       This is (Jacob) Bernoulli's inequality. Where did you use \( x \geq -1 \)? Is it really necessary?

5) Define a sequence inductively as follows: \( a_1 = 1, \quad a_{n+1} = \sqrt{1 + a_n} \).
    Show that \( L = \lim_{n \to \infty} a_n \) exists and find a formula for the limit. Answer: The limit is the golden ratio:
    \[ L = \frac{1 + \sqrt{5}}{2} \]
    Hint: First show that \( 0 < a_n < 2 \) for all \( n \), by induction. Then show that the sequence is strictly increasing.
    Then note that \( L \) must satisfy \( L^2 = 1 + L \). Why?
6) State whether the following sequences have limits. If they do, find the limit and prove that the sequence
approaches that limit using the definition of limit. If they don’t, explain why they don’t.

a) \( \lim_{n \to \infty} \sin(n\pi) \); b) \( \lim_{n \to \infty} \frac{n}{n^2 + 1} \); c) \( \lim_{n \to \infty} \frac{n^2 - n}{n^2 + 1} \); d) \( \lim_{n \to \infty} \frac{1}{2^n} \).

7) True-False.
Tell whether the following statements are true or false. Give a brief reason for your answer.

a) The only real number \( a \) which satisfies the inequality \( |a| < \varepsilon \), for every \( \varepsilon > 0 \), is the number \( a=0 \).

b) The sum of two irrational numbers is irrational.

c) For all real numbers \( x,y \), we have \( |x-y| \leq |x| - |y| \).

d) If \( \lfloor x \rfloor \) denotes the largest integer less than or equal to \( x \), then \( \lim_{x \to 1} \lfloor x \rfloor = 1 \).

e) Any decreasing sequence of real numbers has a limit.

f) If \( f:S \to T \) and \( A \) is a subset of set \( T \), define the inverse image \( f^{-1}(A) = \{ x \in S \text{ and } f(x) \in A \} \). Then \( f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B) \) for any subsets \( A \) and \( B \) of the set \( T \). We do not assume here that \( f \) is 1-1 and onto.

g) If \( f:S \to T \) is 1-1 and onto then there is an inverse function \( g:T \to S \) such that the composition \( f \circ g = \text{identity on } T \) and \( g \circ f = \text{identity on } S \).

h) If \( x>0 \) and \( y \) is any real number, there is a positive integer \( n \) such that \( nx>y \).

i) A sequence of real numbers can have 2 different limits.

8. Find the lub and glb for the following sets, if possible.

a) \( \left\{ \frac{m}{2m+1} \mid m \in \mathbb{Z}^+ \right\} \);

b) \( \left\{ x \in \mathbb{Q} \mid x > 0 \text{ and } x^2 < 5 \right\} \);

c) the positive integers

d) the open interval \( (0,3) \).

9) Find the following limits, if possible. Then explain your answer using the definition of limit.

a) \( \lim_{x \to 0} \sqrt{x} \);

b) Define \( f(x) = 0 \) if \( x \) is irrational and \( f(x) = 1 \) if \( x \) is rational. Find \( \lim_{x \to a} f(x) \) for any rational number \( a \).