(a) Lecture Notes, pp. 8-9

Examples

function \( f : \mathbb{R} \to \mathbb{R} \), \( f(x) = x^2 \quad \forall x \in \mathbb{R} \)

1-1 function \( f : \mathbb{R}^+ \to \mathbb{R} \), \( f(x) = x^2 \quad \forall x \in (0, \infty) \)

onto function \( f : \mathbb{R}^+ \to \mathbb{R}^+ \), \( f(x) = x^2 \quad \forall x \in (0, \infty) \)

(b) Lecture Notes, p. 13

Example: \( \mathbb{Z}^+ \)

(c) Lecture Notes, p. 25

Example: l.u.b. \( \{ x \mid x^2 < 2 \} = \sqrt{2} \)

(d) Lecture Notes, p. 27

Example: \( \lim_{n \to \infty} \frac{1}{n} = 0 \)

(e) Lecture Notes, p. 39

Example: \( \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2 \)

(f) Lecture Notes, p. 35

Example: \( \{ f_n \}_{n=1}^\infty \) is a Cauchy sequence

(g) Lecture Notes, p. 36

Example: sequence \( x_n = (-1)^n \), \( n \in \mathbb{Z}^+ \)

subsequence \( y_k = x_{2k} = (-1)^{2k} = 1 \), \( k \in \mathbb{Z}^+ \)
2a) Notes, p. 10
2b) Notes, pp. 22-23
2c) Notes, p. 25
2d) \[ \forall x \in \mathbb{R}, \, \exists \varepsilon > 0 \text{ s.t. } \forall y > 0 \, \exists \delta > 0 \text{ s.t. } |x - a| < \delta \text{ and } |f(x) - f(a)| < \varepsilon \]

3a) Notes, pp. 29-30
3b) Notes, pp. 36-37
3c) Notes, pp. 36, 38
3d) Notes, p. 31, Lemma 2
3e) Homework 3, Problem 3

4a) Proof by Contradiction: \( \sqrt{5} = \frac{m}{n} \) with \( m, n \in \mathbb{Z}^+ \), \( m, n \) having no common integer divisor but \( \pm 1 \), i.e. \( \frac{m}{n} \) is in lowest terms.

(Square) \( 5 = \frac{m^2}{n^2} \Rightarrow 5n^2 = m^2 \)

\( \Rightarrow m = 5k \text{ for some } k \in \mathbb{Z} \)

Why? otherwise \( m = 5k + r \), \( r \in \{1, 2, 3, 4\} \).
\[ m = \left( 5k + r \right)^{\frac{-3}{2}}, \quad r \in \{1, 2, 3, 4\} \]

\[
\begin{align*}
    m^2 &= (5k + 1)^2 = 25k^2 + 10k + 1 = 5k^2 + 1 \\
    m^2 &= (5k + 2)^2 = 25k^2 + 20k + 4 = 5k^2 + 4 \\
    m^2 &= (5k + 3)^2 = 25k^2 + 30k + 9 = 5k^2 + 9 \\
    m^2 &= (5k + 4)^2 = 25k^2 + 40k + 16 = 5k^2 + 16
\end{align*}
\]

But \( m^2 = 5n^2 \) contradicts any of these.

So \( m = 5l \Rightarrow m^2 = 25l^2 = 5n^2 \Rightarrow 5k^2 = n^2 \Rightarrow n = 5l \).

But this contradicts \( m/n \) in lowest terms.

\[ Q = \bigcup \left\{ \frac{m}{n} \mid m, n \in \mathbb{Z} \right\} \]

By Prop. 4.6 in Lang, p. 14, we need only show

\( \mathbb{Z} \) denumerable. But \( \mathbb{Z} = \mathbb{Z}^+ \cup 0 \cup -\mathbb{Z}^+ \),

\( \mathbb{Z}^+ \) is denumerable,

\( \phi : \mathbb{Z}^+ \rightarrow -\mathbb{Z}^+ \) is 1-1, onto \( \Rightarrow -\mathbb{Z}^+ \) denumerable.

The union of a denumerable set and a finite set

is denumerable. See Lecture Notes p. 17.

The union of denumerable sets is denumerable.

Notes, p. 17.

Thus: \( \mathbb{Z} \) is denumerable.

This can be proved by Cantor's diagonal argument

on pp. 18-19 of the notes. If you represent

real numbers using base 10, you see that

the numbers in \( [0, 1] \)

\[ \text{(decimal representation)} \quad \sum_{j=0}^{\infty} q_j 10^{-j}, \quad q_j \in \{0, 1, \ldots, 9\} \]

form a non-denumerable set. As a subset

of \( \mathbb{R} \) is non-denumerable, \( \mathbb{R} \) must be

non-denumerable. See Cor. 4.2 of Lang, p. 13.

You can think of this as an infinite vector whose

entries are elements of \( \{0, 1, 2, \ldots, 9\} \). The diagonal

argument works as it would with 0's + 1's.
Note: \( 0.9999\ldots = 1 \)
causes some real numbers to have 2
different decimal representations.
This doesn't really change our argument.
Even if we only look at real numbers
in \((0,1)\) whose representation by decimals
involves only \(0\)'s and \(1\)'s or \(4\)’s and \(5\)'s
we still have a non-denumerable set.

\[ \mathcal{S} \rightarrow \mathcal{T} \quad \mathcal{S} = \{ s_1, \ldots, s_n \} \]
\[ \mathcal{T} = \{ t_1, \ldots, t_k \} \]

\( \mathcal{f} \) is completely determined by the
\( n \)-tuple
\((f(s_1), f(s_2), \ldots, f(s_n)) \in \mathcal{T} \times \mathcal{T} \times \cdots \times \mathcal{T} = \mathcal{T}^n. \)
So we have a \( 1 \)-1 onto map
\( \mathcal{f}: \mathcal{S} \rightarrow \mathcal{T} \leftrightarrow \mathcal{T}^n \)
Thus \( |\mathcal{f}: \mathcal{S} \rightarrow \mathcal{T}| = |\mathcal{T}|^n = |\mathcal{T}| = 151 \)
\[ \text{since} \quad |A \times B| = |A| \cdot |B| \]

\( (1+x)^n \geq 1+nx \quad \text{if } x \geq -1 \quad \forall n \in \mathbb{Z}^+ \)

**Proof by Induction**

\( n=1 \)
\[ 1+x \geq 1+x \quad \checkmark \]

**Induction Step**
\[ \mathcal{I}_n \Rightarrow \mathcal{I}_{n+1} \]
\[ (1+x)^n \geq 1+nx \]
Multiply by \((1+x)\):
\[ (1+x)^{n+1} \geq (1+x)(1+nx) = 1+nx+x+n^2x^2 \]
\( \Rightarrow (1+x)^{n+1} \geq 1+(n+1)x+n^2 \quad \checkmark \)

\[ \text{Note: } 1+x \geq 0 \quad \text{as } x \geq -1 \]
\[ 8 \quad a \leq -1, \quad 1+a < 0 \Rightarrow (1+a)^5 < 1+5a \]
This is false if \( a = -3 \), for example.
\[ as \quad 32 < -14. \]
Claim 1

\[ 0 < a_n < 2, \quad \forall n=1,2,3, \ldots \]

Proof by Induction

1. \[ a_1 = 1 \in (0,2) \]

2. Truth for \( n \) \( \Rightarrow \) truth for \( n+1 \)

\[ 0 < a_n < 2, \quad a_{n+1} = \sqrt{1+a_n} \]

First \( a_{n+1} > 0 \) (positive square root)

Since \( \sqrt{x} \) is an increasing function

by Exercise 2 in Lang, p. 24,

if \( a_n < 2 \), then \( 1+a_n < 3 \) and thus

\[ \sqrt{1+a_n} < \sqrt{3} < \sqrt{4} = 2 \]

\( \Rightarrow \) \( a_{n+1} < 2 \)

Claim 2

\[ a_n < a_{n+1}, \quad \forall n=1,2,3, \ldots \]

Proof by Induction

1. \( a_1 = 1 < a_2 = \sqrt{1+1} = \sqrt{2} \)

   Again \( \sqrt{x} < \sqrt{2} \) since \( \sqrt{x} \) increasing (strictly)

2. Assume \( a_n < a_{n+1} \) and show \( a_{n+1} < a_{n+2} \)

\[ a_{n+1} = \sqrt{1+a_n} \leq \sqrt{1+a_{n+1}} = a_{n+2} \]

(since \( \sqrt{x} \) strictly increasing)

Claim 3

\[ \exists \lim_{n \to \infty} a_n = l, \text{ where } L_n \in \mathbb{Z}^+ \]

Proof

See Notes pp. 28-30.

Claim 4

\[ L = \frac{1+\sqrt{5}}{2} \]

Proof

\[ a_{n+1} = \sqrt{1+a_n} \]

\[ a_{n+1}^2 = 1 + a_n \]

\[ L^2 = 1 + L \]

Here we use properties of limits (Notes p. 29)

\[ L^2 - L - 1 = 0 \]

\[ \Rightarrow \quad L = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2} \]

We must take positive root as limits preserve \( 0 \leq a_n \).
\( \lim_{n \to 0} \sin(n\pi) = \lim_{n \to 0} 0 = 0 \)

\( x_n = 0 \quad \forall n \in \mathbb{Z}^+ \implies |x_n - 0| = 0 < \epsilon \quad \forall n \geq 1 \)

Set \( N_\epsilon = 1 \). Then \( n > N_\epsilon \implies |\sin(n\pi) - 0| < \epsilon \).

\( \lim_{n \to \infty} \frac{n}{n^2+1} = \lim_{n \to \infty} \frac{n^2}{1+n^2} = 0 \)

Given \( \epsilon > 0 \), we want \( N_\epsilon \) so that \( n > N_\epsilon \implies \left| \frac{n}{n^2+1} \right| < \epsilon \).

\( \Rightarrow \quad n < \epsilon \left(\frac{n^2+1}{n^2}\right) \iff \frac{1}{\epsilon} < \frac{n^2+1}{n^2} = \frac{n}{n^2} + \frac{1}{n^2} \)

\( n + \frac{1}{n} > \frac{1}{\epsilon} \implies \text{we should take } N_\epsilon = \left\lceil \frac{1}{\epsilon} \right\rceil \)

Then \( n > \left\lceil \frac{1}{\epsilon} \right\rceil \) \( \implies \left| \frac{n}{n^2+1} - 0 \right| < \epsilon \)

\( \lim_{n \to \infty} \frac{n^2-n}{n^2+1} = \lim_{n \to \infty} \left( \frac{n^2-n}{n^2+1} \right) = \lim_{n \to \infty} \left( \frac{1-\frac{1}{n}}{1+\frac{1}{n^2}} \right) = 1 \)

Given \( \epsilon > 0 \), we want to show

\( \left| \frac{n^2-1}{n^2+1} - 1 \right| < \epsilon \)

Well, the inequality above is equivalent to

\( \epsilon > \left| \frac{n^2-n-1}{n^2+1} \right| = \frac{n+1}{n^2+1} \)

\( \iff \quad \frac{n^2+1}{n+1} \geq \frac{1}{\epsilon} \)

\( \frac{n^2+1}{n+1} > \frac{n^2}{2n} = \frac{n}{2} \)

\( \frac{n}{2} > \frac{1}{\epsilon} \iff n > \frac{2}{\epsilon} \). So let \( N_\epsilon = \left\lceil \frac{2}{\epsilon} \right\rceil \)

\( n \geq \left\lceil \frac{2}{\epsilon} \right\rceil = N_\epsilon \implies \epsilon > \left| \frac{n^2-n}{n^2+1} - 1 \right| \)

retracing the above inequalities
(6d) \[
\lim_{n \to \infty} \frac{1}{2^n} = 0
\]

Given \( \varepsilon > 0 \) we must find \( N_\varepsilon \) s.t. \( n > N_\varepsilon \implies \frac{1}{2^n} < \varepsilon \) \( \frac{1}{2^n} < \varepsilon \iff 2^n > \frac{1}{\varepsilon} \)
\( 2^n > n \) for \( n \geq 1 \)

Proof by induction
\( n=1 \quad 2^1 > 1 \quad \checkmark \)
Assume for \( n \) holds prove for \( n+1 \)
\( \frac{2^n}{2} > n \)
Multiply by \( 2 \)
\( 2 \cdot 2^n > 2n \)
\( \implies 2^{n+1} > 2n = n+n > n+1 \quad \forall \quad n \geq 1 \)

So if \( N_\varepsilon = \left\lfloor \frac{1}{\varepsilon} \right\rfloor \) we have
\( n > N_\varepsilon \implies \frac{1}{2^n} > \frac{1}{\varepsilon} \)
\( \implies \left| \frac{1}{2^n} - 0 \right| = 2^{-n} < \varepsilon \) QED

(7) a.) True
otherwise \( |a| > 0 \), take \( \varepsilon = \frac{|a|}{2} \). Then
\( |a| < \frac{|a|}{2} \implies 1 < \frac{1}{2} \iff 2 < 1 \iff 1 < 0 \)

But \( 1 > 0 \) otherwise \(-1 > 0 \) and \((-1)(-1) = 1 > 0 \) contradicting the positive property of \( \mathbb{Q} \)

b) False
\( \frac{1}{\sqrt{a}} + (\sqrt{a}) = 0 \in \mathbb{Q} \) but \( \sqrt{2}, -\sqrt{2} \notin \mathbb{Q} \)

c) False
\( 3 = |12 - 5| \leq |12| - |5| = -3 \) is false

d) False
\( \lim_{x \to 1^+} 1/x = 1 \neq 0 = \lim_{x \to 1^-} 1/x \)
1e) False
\[ a_n = -n \] is decreasing with no finite limit

2\( \) True
\[ x \in f^{-1}(A \cap B) \iff f^{-1}(x) \in A \cap B \]
\[ \iff f^{-1}(x) \in A \text{ and } f^{-1}(x) \in B \]
\[ \iff x \in f^{-1}(A) \text{ and } x \in f^{-1}(B) \]
\[ \iff x \in f^{-1}(A) \cap f^{-1}(B) \]

3\( \) True
Define for \( y \in T \), \( g(y) = \frac{e^y}{5} \)
\[ \exists x \text{ since } f \text{ is onto.} \]
\[ x \text{ is unique since } f \text{ is 1-1.} \]
Also \( g(f(x)) = g(y) = x \)
\[ f(g(y)) = f(x) = y \]
\[ \forall x \in S, \forall y \in T \]

4\( \) True
This is the Archimedean property of \( \mathbb{Z}^+ \)
Otherwise \( \mathbb{Z}^+ \) is bounded above by \( y/x \).

5\( \) False
\[ \lim_{n \to \infty} x_n = L \text{ and } \lim_{n \to \infty} x_0 = M \]
\[ \Rightarrow |L - M| = |L - x_n + x_n - M| \]
\[ \leq |L - x_n| + |x_n - M| \]
\[ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \text{as } n \to \infty \]
\[ 0 \quad 0 \quad 0 \]
So \( |L - M| = 0 \) as limits preserve \( \leq \)
and \( 0 \leq |L - M| \leq 0 \).
\(\textbf{b) g.l.b.} = 0\)

\(\text{l.u.b.} = \sqrt{5}\) as \((x^2 < 5)\) \(\Rightarrow 0 < x < \sqrt{5}\)\n
\(\text{strictly increasing}\)

\(\text{c) There is no l.u.b. for } \mathbb{Z}^+ \text{ as it is not bounded above.}\)

\(\text{The g.l.b. of } \mathbb{Z}^+ \text{ is } 1.\)

\(\text{d) l.u.b. } = 3 \quad \text{g.l.b. } = 0\)
\[ (a) \quad \lim_{x \to 0} \sqrt{x} = 0 \]

\[ \forall \varepsilon > 0 \text{ we must find } \delta > 0 \text{ s.t. } |x - 0| < \delta \Rightarrow |\sqrt{x} - 0| < \varepsilon \]

\[ (\sqrt{x} < \varepsilon) \Leftrightarrow 0 < x < \varepsilon^2 \]

Let \( \delta = \varepsilon^2 \). Then \( 0 < x < \delta = \varepsilon^2 \Rightarrow \sqrt{x} < \varepsilon \)

\[ (b) \quad \text{if } a \in \mathbb{Q} \text{ then } \lim_{x \to a} f(x) \text{ does not exist} \]

\[ \text{Why?} \quad \left\{ \begin{array}{l} \forall \delta > 0 \text{ there are both rational \&} \\
\text{irrational numbers in } (a - \delta, a + \delta) \\
\text{So there are } x, x' \in (a - \delta, a + \delta) \text{ s.t.} \\
f(x) = 0, \quad f(x') = 1 \\
\end{array} \right. \]

If we let \( \varepsilon < \frac{1}{2} \) then

\[ |f(x) - L| < \varepsilon, \quad |f(x') - L| < \varepsilon \Rightarrow \]

\[ |x - x'| < \delta \]

\[ |f(x) - f(x')| = |f(x) - L + L - f(x')| \leq |f(x) - L| + |f(x') - L| < 2\varepsilon = 1 \]

\[ \Rightarrow |x - x'| < \delta \text{ contradiction!} \]

\[ \text{To see why, all intervals contain rationals as well as irrationals,} \]

\[ \text{see Lang, p.31, Prop.4.3 and} \]

\[ \text{Lang, p.32, Exercise 2.} \]