1) a) Define and give an example of a function $f(x)$ which is continuous at $x=c$.
b) Give 2 definitions for the derivative $f'(c)$. Draw sketches to illustrate the meaning of both defns.
c) Define $a^x$ for $a>0$ and explain why this generalizes $a^n$ for $n \in \mathbb{Z}^+$.
d) State the 2 axioms for the integral of a continuous function on $[a,b]$.

2) State and prove:
   a) mean value theorem             b) fundamental theorem of calculus  
   c) Weierstrass theorem on the existence of maxima and minima  d) intermediate value theorem

3) Prove the following properties of the derivative
   a) chain rule                     b) linearity                    c) product rule

4) Prove the following properties of the integral of a continuous function:
   a) $\int_{a}^{b} f'(x)dx = f(b) - f(a)$, assuming $f'$ continuous on $[a,b]$. b) linearity
   c) integration by parts         d) substitution formula  e) integrals preserve $\leq$

5) True-False.
   Tell whether the following statements are true or false. If true, give a brief reason for your answer. If false, give a counterexample.
   a) Suppose $f:(a,b) \to \mathbb{R}$ is continuous. Then there is a point $c$ in $(a,b)$ such that $f(x) \leq f(c)$ for all $x \in (a,b)$.
   b) $\int_{a}^{b} fg = \int_{a}^{b} f \int_{a}^{b} g$ for functions $f,g:[a,b] \to \mathbb{R}$ continuous.
   c) Suppose $f:[a,b] \to \mathbb{R}$ is continuous and $f$ is differentiable on $(a,b)$ with $f'(x) = 0$ for all $x \in (a,b)$. Then $f(x) =$ constant, for all $x \in [a,b]$.
   d) Recall that a function $f(x)$ defined for $x \in \mathbb{R}$ is even iff $f(-x) = f(x)$ for all $x \in \mathbb{R}$. We say $f(x)$ is odd iff $f(-x) = -f(x)$, for all $x \in \mathbb{R}$. Then the derivative $f'(x)$ of an even function $f(x)$ is an odd function.
   e) Suppose that $x = g(y)$ is the inverse function to $y = f(x)$. Then $g'(y) = 1/f'(y)$.
   f) $f, g$ continuous on $[a,b]$ implies $\max\{f, g\}$ continuous on $[a,b]$.
   g) $f$ continuous at $x = c$ implies $f$ differentiable at $x = c$. 


6) a) Compute \( g'(21) \) when \( x=g(y) \) is the inverse function for \( y=f(x)=2x^3+1 \).

b) Find the set of points where the following function \( f(x) \) is continuous:

\[
f(x) = \begin{cases} 
1, & x \text{ rational} \\
0, & x \text{ irrational}
\end{cases}
\]

c) Compute \( \lim_{x \to \infty} \left( 1 + \frac{x}{n} \right)^n \).

d) Suppose that \( f \) is a differentiable function on the whole real line such that \( f'(x)=2xf(x) \) for all \( x \in \mathbb{R} \).

Show that there is a constant \( C \) so that \( f(x) = Ce^{x^2} \) for all \( x \).

e) Compute \( \frac{d}{dt} \int_0^t g(x-t)dx \).

7) a) Suppose that \( f'(x) > 0 \) for all \( x \) in \((a,b)\). Why does \( y=f(x) \) have an inverse function \( x=g(y) \)? What is the formula for the derivative of the inverse function? Prove it.

b) State Taylor’s formula with remainder (2nd version). Apply it to get the formula for \( \log(1-x) \) using the first 3 terms of the Taylor series plus remainder.

c) Define \( e^x \) by its Taylor series and prove that \( e^x e^y = e^{x+y} \).

d) Define \( \log(y) \) as the inverse function to \( e^x \) and show that \( \log(uy) = \log u + \log v \).

8) Define a function \( f(x) = \begin{cases} 
e^{-1/x}, & \text{for } x > 0 \\
0, & \text{for } x \leq 0
\end{cases} \). Show that \( f^{(n)}(0) \) exists for all \( n \in \mathbb{Z}^+ \). Sketch this function. Is \( f(x) \) represented by its Taylor series around the point 0? Why?

9) Show that the following function gives an example of a continuous function whose graph cannot be drawn without lifting pen from paper. Define \( f(x) \) inductively as a map from \([-1,1]\) to \([-1,1]\) consisting of straight line segments connecting the point \((1/(2k),0)\) to the 2 points \((1/(2k+1),1/(2k+1))\) and \((1/(2k-1),1/(2k-1))\) for \( k=\pm1,\pm2,\pm3, \ldots \). Sketch the function.