1) Suppose that $\langle \cdot, \cdot \rangle$ is a scalar product on a vector space $V$ and suppose that we have a subset $S$ of $V$ with a vector $a$ adherent to $S$. Let $f$ and $g$ be functions mapping $S$ into $V$ such that the two following limits exist
$$\lim_{x \to a} f(x) = L \quad \text{and} \quad \lim_{x \to a} g(x) = M.$$ Prove that then $\lim_{x \to a} \langle f(x), g(x) \rangle = \langle L, M \rangle$.

2) Define $C[a,b]$ to be the space of continuous real valued functions on the finite interval $[a,b]$. Define the mapping $I: C[a,b] \to \mathbb{R}$ by $I(f) = \int_a^b f(x)\,dx$ for $f \in C[a,b]$.

   a) Is $I$ continuous when we use the $\| \cdot \|_\infty$ norm on $C[a,b]$ and ordinary absolute value as our norm on $\mathbb{R}$? Why?
   
   b) What if we use the $\| \cdot \|_1$ norm on $C[a,b]$? Why?

3) Consider the following functions on $\mathbb{R}^2$
$$f(x,y) = \begin{cases} \frac{2x^2y}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases} \quad g(x,y) = \begin{cases} \frac{(y^2-x^2)^2}{x^4+y^4}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

   a) Does $\lim_{x \to 0} \lim_{y \to 0} h(x,y) = \lim_{y \to 0} \lim_{x \to 0} h(x,y) = L$ when $h = f$ or $h = g$?

   b) Do the repeated limits in part a) equal the limit as $(x,y) \to (0,0)$ using any of the favorite norms on $\mathbb{R}^2$? That is, does $\lim_{(x,y) \to (0,0)} h(x,y) = L$?

   c) Are either of the functions $f, g$ continuous at the origin?

4) Let $\ell^2$ denote the set of all sequences $a=\{a_n\}_{n=1}^\infty$ of real numbers such that $\sum_{n=1}^\infty a_n^2$ converges.

   a) Show that you can define addition and scalar multiplication to make $\ell^2$ a vector space.

   b) Define for $a=\{a_n\}_{n=1}^\infty$ and $b=\{b_n\}_{n=1}^\infty$ in $\ell^2$, $\langle a, b \rangle = \sum_{n=1}^\infty a_n b_n$. Show that this series converges and has the properties of a scalar product. You will need to use the Cauchy-Schwarz inequality on the partial sums.