1) \( x \in A \implies \forall \delta > 0 \exists a \in A \text{ s.t. } ||a-x|| < \delta \).

Since \( a \in A \subset B \) this \( \implies a \in B \) and so \( x \in B \).

2) Proof by Picture: \( A = \{ (x, y) \in \mathbb{R}^2 | x < y \} \)
   
   If \( x < y \) then \( (x, y) \in A \)
   
   If \( x = y \), then \( \forall \delta > 0 \)
   
   \((x, x) - (x, x)\) \in A s.t.
   
   Every point of \( A \) is distance \( r \) to \((p, q)\).
   
   Every point of \( A \) is distance \( r \) to \((p, q)\).
   
   So a circle about \((p, q)\) of radius \( r \) has no points of \( A \).
   
   Thus such points \((p, q)\) are not in the closure of \( A \). It follows that the closure of \( A \) is the points \((x, y)\) with \( x \leq y \).

3) \( f(x) = x^2 \).

\[
\int_0^1 x^2 \, dx = 0.1(\sqrt{1}-0) + 0.2(\sqrt{2}-\sqrt{1}) + 0.3(\sqrt{3}-\sqrt{2}) + 0.4(\sqrt{4}-\sqrt{3}) + 0.5(\sqrt{5}-\sqrt{4}) + 0.6(\sqrt{6}-\sqrt{5}) + 0.7(\sqrt{7}-\sqrt{6}) + 0.8(\sqrt{8}-\sqrt{7}) + 0.9(\sqrt{9}-\sqrt{8}) + 1(1-\sqrt{9})
\]
4) \( f(x) = x \sin \frac{1}{x}, \quad x \neq 0 \), \( f(0) = 0 \)

\[ g(x) = 1 \quad \text{if} \quad x > 0, \quad g(x) = -1 \quad \text{if} \quad x < 0 \]

\( h(x) = g(f(x)) \) is a uniform limit of a sequence of step functions on \([0,1]\).

The idea is that near \( x=0 \) a step function will be constant while \( h \) will be alternating madly between \(-1\) and \(+1\).

So we should find \( \| s - h \|_\infty \geq 1 \) for any step function \( s \).

More precisely, on any interval \((a, b)\), suppose \( s(x) = c \) for all \( x \in (a, b) \).

As \( x \to 0^+ \), \( \frac{1}{x} \to \infty \). So \( \exists \alpha, \beta \in (a, b) \)

\[ \sin \left( \frac{1}{\alpha} \right) = 1, \quad \sin \left( \frac{1}{\beta} \right) = -1 \]

Then \( s(\alpha) = 1 \) and \( s(\beta) = -1 \).

So \( \| s - h \|_\infty \geq \max \{ |s(\alpha) - h(\alpha)|, |s(\beta) - h(\beta)| \} \)

\[ = \max \{ |c - 1|, |c + 1| \} \geq 1 \]

\[ \begin{cases} \frac{1}{x} & x > 0 \\ \frac{1}{-x} & x < 0 \end{cases} \]

The max \( c \) would be 1, but \( \boxed{a = 1 - \frac{1}{1 + 1} < 1} \)

see picture