1) a) **A Mean Value Theorem.** Suppose that \( f, g : [a, b] \to \mathbb{R} \) are continuous on \([a, b]\) and \( g(x) \geq 0 \) for all \( x \in [a, b] \). Show that there exists \( c \in [a, b] \) such that
\[
\int_a^b fg = f(c)\int_a^b g.
\]

b) Show that the result of problem 1 need not be true if \( g(x) \) can assume both positive and negative values on \([a, b]\).

2) Define \( \int_a^b f = - \int_b^a f \). Show that \( \int_a^b f = \int_a^c f + \int_c^b f \) no matter what the ordering of the points \( a, b, c \) is. There are 6 possible orderings starting with \( a < c < b \), \( a < b < c \), etc.

3) Let \( g : [a, b] \to \mathbb{R} \) be continuous and \( g(x) \geq 0 \) for all \( x \in [a, b] \). Show that if \( \int_a^b g = 0 \) then \( g(x) = 0 \ \forall x \in [a, b] \).

4) Let \( f : [0, 1] \to \mathbb{R} \) be defined by \( f(x) = 1 \) for all \( x \) in \([0, 1]\). Define \( F \) by
\[
F(x) = \begin{cases} 
1, & x = 0 \\
0, & x = 1 \\
x, & 0 < x < 1
\end{cases}
\]

Clearly \( F'(x) = f(x) \) for all \( x \in (0, 1) \). However \( F(1) - F(0) \neq \int_0^1 f \).

What went wrong with the fundamental theorem of calculus here?