1) Since \( f \) is continuous on \([a, b]\), Weierstrass tells us it attains a max \( M \) and a min \( m \).
So \( m \leq f(x) \leq M \) \( \forall x \in [a, b] \).
Since \( g(x) > 0 \), this \( \Rightarrow \)
\[ mg(x) \leq (fg)(x) \leq Mg(x) \quad \forall x \in [a, b], \]
Since integrals preserve \( \leq \), this \( \Rightarrow \) \( m \int_a^b g(x) \, dx \leq \int_a^b (fg)(x) \, dx \leq \int_a^b Mg(x) \, dx = M \int_a^b g(x) \, dx \)
using the linearity of the integral.
(We know \( f, g \) continuous on \([a, b]\) \( \Rightarrow \) \( fg \) continuous on \([a, b]\).
It follows that all the functions we are trying to integrate are really integrable.)
If \( \int_a^b g = 0 \) then by \( \bigstar \) \( \int_a^b fg = 0 \)
and our mean value thm is true.
So assume \( \int_a^b g \neq 0 \), so \( \int_a^b g > 0 \) as
integrals preserve \( \geq \). Thus we can divide \( \bigstar \) by \( \int_a^b g \) and get
\[ m \leq \frac{\int_a^b fg}{\int_a^b g} \leq M \]
Now use the intermediate value thm to see \( \exists c \in (a, b) \) s.t.
\[ \frac{\int_a^b fg}{\int_a^b g} = f(c) \]
This \( \Rightarrow \) our mean value thm.
b) Suppose \([a, b] = [-1, 1]\) and \(g(x) = x = f(x)\).

Then
\[
\int_a^b f \, dx = \int_a^1 x^2 \, dx = \frac{x^3}{3} \bigg|_1^1 = \frac{2}{3}
\]
\[
\int_a^b g \, dx = \int_a^1 x \, dx = \frac{x^2}{2} \bigg|_1^1 = 0
\]

Our mean value thm \(\Rightarrow\)
\[
\frac{2}{b-a} = \frac{\int_a^b f \, dx}{\int_a^b g \, dx} = f(c) \int_a^b g \, dx = 0 \Rightarrow \frac{2}{b-a} = 0
\]
impossible

(1) \(a \leq c \leq b \Rightarrow \int_a^b f \, dx = \int_a^c f \, dx + \int_c^b f \, dx\) an axiom for integrals.

(23) \(a \leq b \leq c \Rightarrow \int_a^c f \, dx + \int_b^c f \, dx = (\int_a^c f \, dx + \int_b^c f \, dx) + \int_b^c f \, dx
\]
\[
= \int_a^c f \, dx + \int_b^c f \, dx - \int_c^b f \, dx = \int_a^b f \, dx
\]

(12) \(c \leq a \leq b \Rightarrow \int_a^c f \, dx + \int_b^c f \, dx = -\int_c^a f \, dx + \int_c^b f \, dx
\]
\[
= \int_a^b f \, dx
\]

(13) \(b \leq c \leq a \Rightarrow \int_a^c f \, dx + \int_b^c f \, dx = -\int_c^b f \, dx - \int_c^a f \, dx
\]
\[
= -\int_c^a f \, dx + \int_b^c f \, dx = \int_a^b f \, dx
\]

(123) \(b \leq a \leq c \Rightarrow \int_a^c f \, dx + \int_b^c f \, dx = \int_a^c f \, dx - \int_b^c f \, dx
\]
\[
= \int_a^c f \, dx - (\int_b^c f \, dx + \int_a^c f \, dx)
\]
\[
= -\int_a^c f \, dx + \int_b^c f \, dx = \int_a^b f \, dx
\]

(132) \(c \leq b \leq a \Rightarrow \int_a^c f \, dx + \int_b^c f \, dx = -\int_c^b f \, dx + \int_c^a f \, dx
\]
\[
= -\int_c^b f \, dx + \int_b^c f \, dx - \int_c^a f \, dx = \int_a^b f \, dx
\]
3) Proof by Contradiction

Suppose \( g(c) > 0 \) for \( c \in [a, b] \), \( \exists \delta > 0 \)

since \( g \) continuous on \( [a, b] \).

But then \( \frac{g(c)}{2} > g(x) - g(c) > -\frac{g(c)}{2} \)

\[ \Rightarrow g(x) > \frac{g(c)}{2} \quad \forall x \in [a, b] \cap (c-\delta, c+\delta) \]

\[ \Rightarrow 0 = \int_{a}^{b} g(x) \, dx \geq \frac{g(c)}{2} \cdot \delta > 0 \]

This is a contradiction

So we're done

4) \( F(1) - F(0) = -1 \)

\[ \int_{0}^{1} x \, dx = \left[ \frac{x^2}{2} \right]_{0}^{1} = \frac{1}{2} \neq -1 = F(1) - F(0) \]

The problem is \( f(x) \) is not continuous on \( [0, 1] \) and we needed that to prove the fundamental thm of calculus