Lecture 3. Limits of Functions and Continuity

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1 Limits of Functions

Notes.

I am skipping the last section of Chapter 6 of Lang; the section about open and closed sets. We can probably live without more definitions (unless you plan to go to grad school in math.). Briefly an open set in the plane has a fuzzy boundary like the open disc \( \{ x \mid \|x\| < 1 \} \). A closed set is the complement of an open set. So the boundary is hard not fuzzy; e.g., like the closed disc \( \{ x \mid \|x\| \leq 1 \} \). You can use open sets to eliminate \( \varepsilon \delta \) and limit from the definition of continuous function. See Lang, Theorem 5.6, p. 156. See Figure 1 for pictures of open and closed sets.

So that brings us to Lang, Chapter 7 - the story of limits of functions in and on normed vector spaces. Why are we interested in this question? We want to know when we can interchange limit and integral, limit and derivative. We want to know whether a series of functions such as a Taylor series or a Fourier series converges. In fact, we need to know precisely what we mean by convergence of a series of functions. We want to think of the definite integral as a function on the space \( C[a;b] \) of continuous functions on a finite closed interval \( [a;b] \). What sort of function is it? Linear? Continuous?

In fact proving things about limits of functions into/on normed vector spaces is no harder than it was for functions into/on the real line. We will basically copy the proofs from Lecture 3 of Math. 142a. Suppose that \( V \) and \( W \) are normed vector spaces. I will use the same symbol for the norm on \( V \) and the norm on \( W \). Suppose \( S \subset V \) and \( f : S \to W \). Before defining the limit \( \lim_{x \to a} f(x) = L \) in \( W \) to mean that for every \( \varepsilon > 0 \), there exists \( \delta > 0 \) (depending on \( \varepsilon \)) such that \( x \in S \) and \( \|x - a\| < \delta \) implies \( \|f(x) - L\| < \varepsilon \).

Definition 1 Let \( V \) be a normed vector space with norm \( \|\| \) and \( S \subset V \). We say the point \( a \in V \) is adherent to \( S \) iff for every \( \delta > 0 \) there is a point \( x \in S \) such that \( \|x - a\| < \delta \).

You should picture an adherent point to a set as a point sticking to the set. See Figure 2.

Examples.

1) Any point in \( S \) is adherent to \( S \).
2) If \( V = \mathbb{R}^2 \), using any of our favorite norms, any point in the closed ball \( \{ x \in \mathbb{R}^2 \mid \|x\| \leq r \} \) is adherent to the open ball \( S = \{ x \in \mathbb{R}^2 \mid \|x\| < r \} \).

Definition 2 Suppose \( S \subset V \) and \( f : S \to W \). Here \( V \) and \( W \) are normed vector spaces. We will denote both norms by \( \|\| \) though they may be different. Assume \( a \in V \) is adherent to \( S \). Define \( \lim_{x \to a} f(x) = L \in W \) to mean that for every \( \varepsilon > 0 \),

\[
\lim_{x \to a} f(x) = L \in W \text{ to mean that for every } \varepsilon > 0, \quad x \to a \quad x \in S
\]

there exists \( \delta > 0 \) (depending on \( \varepsilon \)) such that \( x \in S \) and \( \|x - a\| < \delta \) implies \( \|f(x) - L\| < \varepsilon \).

Examples.

1) Let \( V = C[a,b] \), the space of continuous real-valued functions on the finite interval \( [a,b] \). Define \( I : V \to \mathbb{R} \) by

\[
I(f) = \int_a^b f(x)dx.
\]

Suppose our norms are \( \|f\|_1 = \int_a^b |f(x)|dx \) on \( V \) and the usual absolute value on \( \mathbb{R} \). Does \( \lim_{f \to 0} I(f) = 0 \)?

Answer: Yes. In fact, here \( \delta = \varepsilon \), since using the fact that integrals preserve inequalities, we see that
Figure 1: Pictures of an open set (top) and a closed set (bottom).

Figure 2: Picture of a red adherent point to a purple set.
\[ \|f - 0\|_1 < \varepsilon \quad \text{implies} \quad |I(f) - I(0)| \leq \int_a^b |f(x)| \, dx = \|f\|_1 < \varepsilon. \]

2) Set \( f(x, y) = \frac{y^2 - x^2}{y^2 + x^2}, \) for \( (x, y) \neq (0, 0) \) in \( \mathbb{R}^2. \) Does \( \lim_{(x, y) \to (0, 0)} f(x, y) = 0 ? \) Here you can use any of our favorite norms on \( \mathbb{R}^2 \) and ordinary absolute value on \( \mathbb{R}. \) Figure 1 below shows the graph of \( z = f(x, y). \)

Answer. No. Set \( y = kx. \) Then \( f(x, kx) = \frac{k^2 - 1}{k^2 + 1}. \) Thus \( f(x, y) \) has a different value on various lines through the origin. In particular, it is \(-1\) on the \( x\)-axis and \(+1\) on the \( y\)-axis. There is no limit as \( (x, y) \) approaches the origin.

There are more examples in homework 2. The properties of limits are similar to those stated in part 3 of the lectures from Math 142A.

Properties of Limits in Normed Vector Spaces.
We assume that \( V, W \) are normed vector spaces, \( a \) is adherent to \( S \subset V, \) \( f, g : S \to W. \)

1) Uniqueness.
Suppose \( \lim_{x \to a} f(x) = L \) and \( \lim_{x \to a} f(x) = M. \) Then \( L = M. \)
\[ \lim_{x \in S} f(x) = L \quad \text{and} \quad \lim_{x \in S} f(x) = M. \]

2) Linearity.
Suppose \( \alpha, \beta \in \mathbb{R} \) and \( \lim_{x \to a} f(x) = L \) and \( \lim_{x \to a} g(x) = M. \) Then \( \lim_{x \to a} (\alpha f(x) + \beta g(x)) = \alpha L + \beta M. \)
\[ \lim_{x \in S} f(x) = L \quad \text{and} \quad \lim_{x \in S} g(x) = M. \]

3) Composite.
Suppose we have 3 normed vector spaces \( V, W, Z \) and functions \( f : S \to T, \) \( g : T \to Z, \) plus we know that the vector \( a \) is adherent to the set \( S \subset V \) and the vector \( L \) is adherent to the set \( T \subset W. \) If \( \lim_{x \to a} f(x) = L \) and \( \lim_{y \to L} g(y) = M, \)
\[ \lim_{x \in S} f(x) = L \quad \text{and} \quad \lim_{y \in T} g(y) = M, \]
then \( \lim_{x \to a} g(f(x)) = M. \)
\[ \lim_{x \in S} f(x) = L \quad \text{and} \quad \lim_{y \in T} g(y) = M. \]

4) Inequalities.
Suppose \( f, g : S \to \mathbb{R} \) and \( f(x) \leq g(x) \) for all \( x \in S. \) If \( \lim_{x \to a} f(x) = L \) and \( \lim_{x \to a} g(x) = M, \) then \( L \leq M. \)
\[ \lim_{x \in S} f(x) = L \quad \text{and} \quad \lim_{x \in S} g(x) = M, \]

5) Independence of Which Equivalent Norm is Used.
Using equivalent norms on either \( V \) or \( W \) leads to the same definition of \( \lim_{x \to a} f(x) = L. \)
\[ \lim_{x \in S} f(x) = L. \]

Some Proofs.
1) \( \|L - M\| = \|L - f(x) + f(x) - M\| \leq \|L - f(x)\| + \|f(x) - M\|. \)
For any \( \varepsilon > 0 \exists \delta_1 \) such that \( \|x - a\| < \delta_1 \) implies \( \|L - f(x)\| < \frac{\varepsilon}{2}. \)
And \( \exists \delta_2 \) such that \( \|x - a\| < \delta_2 \) implies \( \|f(x) - M\| < \frac{\varepsilon}{2}. \)
Thus taking \( \delta = \min\{\delta_1, \delta_2\} \) we see that \( \|x - a\| < \delta \) implies \( \|L - M\| \leq \|L - f(x)\| + \|f(x) - M\| < \varepsilon. \)
Since \( \varepsilon \) was arbitrary this means \( \|L - M\| = 0 \) (see Lang, p. 30 if you don’t believe this). By the first axiom of norms, this implies \( L - M = 0. \)

2) We know that given \( \varepsilon > 0 \exists \delta_1 \) such that \( \|y - L\| < \delta_1 \) implies \( \|g(y) - M\| < \varepsilon. \)
\[ \|(\alpha f(x) + \beta g(x)) - (\alpha L + \beta M)\| = \|\alpha f(x) - \alpha L + \beta g(x) - \beta M\| \leq \|\alpha f(x) - \alpha L\| + \|\beta g(x) - \beta M\| = |\alpha| \|f(x) - L\| + |\beta| \|g(x) - M\|. \]
Figure 3: Picture of the $h(x)$ values (red circles to the right of 0) on the real axis, along with the limiting value $K$ (a blue star to the left of 0). No way can this happen since the red circles can never get close to the blue star.

3) \( \exists \delta_1 \) such that \( \| x - a \| < \delta_1 \) implies \( \| f(x) - L \| < \frac{\varepsilon}{2(1 + |\alpha|)} \).

And \( \exists \delta_2 \) such that \( \| x - a \| < \delta_2 \) implies \( \| g(x) - M \| < \frac{\varepsilon}{2(1 + |\beta|)} \).

Then let \( \delta = \min\{\delta_1, \delta_2\} \) so that \( \| x - a \| < \delta \) implies

\[
\|(\alpha f(x) + \beta g(x)) - (\alpha L + \beta M)\| \leq |\alpha| \| f(x) - L \| + |\beta| \| g(x) - M \|
\leq \frac{\varepsilon |\alpha|}{2(1 + |\alpha|)} + \frac{\varepsilon |\beta|}{2(1 + |\beta|)} < \varepsilon.
\]

4) Define \( h(x) = g(x) - f(x) \). Then \( h(x) \geq 0 \) for all \( x \in S \). Let \( K = M - L \). If \( K < 0 \), we can get a contradiction. We know from Property 1 of Limits that \( K = M - L = \lim_{x \to a} h(x) \). See Figure 3. The red circles are the values of \( h(x) \) to the right of 0 and the blue star is \( K \), to the left of 0. \( \varepsilon = \frac{|K|}{2} \). Then \( \exists \delta \) such that \( \| x - a \| < \delta \) implies \( |h(x) - K| < \frac{|K|}{2} \).

Then, since \( K \) is negative, \( h(x) - K < -\frac{K}{2} \) and, adding \( K \) to both sides, \( h(x) < \frac{K}{2} < 0 \), a contradiction to \( h(x) \geq 0 \).

We refer you to Lang, p.162-3 for the general story of limits of products. You get the joy of considering a special case in Homework 2, problem 2. There are lots of other cases one could look at; e.g., scalar valued function times vector valued function, matrix valued function times matrix valued function,.......

If you hate \( \varepsilon \delta \) stuff, you will love the following theorem, which allows you to think about limits of sequences instead.

**Theorem 3 Sequential Definition of Limits.** Suppose \( S \subset V \) and \( f: S \to W \), where \( V \) and \( W \) are normed vector spaces. Let \( a \in V \) be adherent to \( S \). Then the existence of \( \lim_{x \to a} f(x) = L \) is equivalent to saying that for every sequence of vectors \( \{x_n\} \) in \( S \) such that \( \lim_{n \to \infty} x_n = a \), we have \( \lim_{n \to \infty} f(x_n) = L \) exists.
Proof. We leave this part as an extra credit exercise.

Proof by Contradiction. Suppose $\exists \{x_n\}$ in $S$ s.t. $\lim_{n\to\infty} x_n = a$, we have $\lim_{n\to\infty} f(x_n) = L$. If, by contradiction, $\lim_{x \to a} f(x)$ does not equal $L$. Then using the rules for negating a statement involving lots of $\forall \exists$, we see that $\exists \varepsilon > 0 \text{ s.t. } \forall n \in \mathbb{Z}^+, \exists x_n \in S \text{ with } \|x_n - a\| < \frac{1}{n} \text{ and } \|f(x_n) - L\| \geq \varepsilon$. Since then $\lim_{n\to\infty} x_n = a$, this is a contradiction to $\lim_{n\to\infty} f(x_n) = L$. 

Maybe we should try to draw a picture of the definition of limit in higher dimensions. The problem is that it is hard to draw the graph of a function unless it maps a subset of the plane into the reals. Here the graph of $z = f(x,y)$ is 3-dimensional. Just plot the points $(x, y, f(x,y))$ in 3-space. Of course in the infinite dimensional case good luck drawing pictures. Even drawing the graph of a function from the plane to the plane requires 4 dimensional pictures. You can still project them down to 2 dimensions as you would in the case of a real valued function of 2 variables. Or you can make a movie of the graph being rotated. Figure 1 shows a 3D graph (drawn using Scientific Workplace) of the function $z = f(x,y) = \frac{y^2-x^2}{y^2+x^2}$ from our earlier example. It is hard to see how badly the function fails to have a limit as $(x, y)$ approaches $(0, 0)$.

![3D plot of $z = f(x,y) = \frac{y^2-x^2}{y^2+x^2}$](image)

2 Continuous Functions

Suppose that $V, W$ are normed vector spaces and $U$ is a subset of $V$.

**Definition 4** $f : U \to W$ is **continuous** at $c \in U$ iff $\lim_{x \to c} f(x) = f(c)$.

When $V = W = \mathbb{R}$, we view continuity to mean that the graph of $y = f(x)$ does not break up at $x = c$. When $V = \mathbb{R}^2$, you can think a similar thing about the surface $z = f(x,y)$ in 3-space. But recalling Figure 1 of the function $f(x,y) = \frac{y^2-x^2}{y^2+x^2}$, it is even hard to see the break up for a function of 2 variables. It is easier to recall that we saw that $f(x,y)$ has a different value on various lines through the origin. It is 1 on the $y$-axis and -1 on the $x$-axis, for example.

**Weird or Perhaps Ridiculous Fact.** Define $c$ to be an isolated point of $U$ to mean that there exists a $\delta > 0$ such that the ball of radius $\delta$ about $c$ contains no points of $U$; i.e. the set $\{x \in U \mid \|x - c\| < \delta\}$ is empty. If $c$ is isolated than any function defined at $c$ is continuous there. If $V = W = \mathbb{R}$, the point $(c, f(c))$ would be disconnected from the rest of the graph of $y = f(x)$. This seems to be a bad choice of terminology, but it appears to be the usual one.

We can use the properties of limits to deduce the following properties of continuous functions.
Properties of Continuous Functions.

1) **Linearity.** Suppose that \( f, g : U \to W \), where \( U \subset V \) and \( V, W \) are normed vector spaces. Let \( \alpha, \beta \) be (real) scalars. Then \( f \) and \( g \) continuous at \( c \in U \) implies that \( (\alpha f + \beta g) \) is continuous at \( c \).

2) **Composition.** Suppose that \( V, W, Z \) are normed vector spaces with \( U \subset V \) and \( T \subset W \). Let \( c \in U \). Suppose that \( f : U \to T \) and \( g : T \to Z \). Suppose \( f \) is continuous at \( c \) and \( g \) is continuous at \( f(c) \). Then \( g \circ f \) is continuous at \( c \).

3) **Sequential Version.**

    Assume \( V, W \) are normed vector spaces with \( U \subset V \). The function \( f : U \to W \) is continuous at \( c \in U \) iff \( \forall \) sequence \( \{v_n\} \) of vectors in \( V \) such that \( \lim_{n \to \infty} v_n = c \), we have \( \lim_{n \to \infty} f(v_n) = f(c) \).

For the proofs, you just have to look at the corresponding properties of limits. We leave it to you and Serge.

**Examples (the same as those in the section on limits).**

1) Let \( V = C[a,b] \), the space of continuous real-valued functions on the finite interval \( [a,b] \). Define \( I : V \to \mathbb{R} \) by

\[
I(f) = \int_a^b f(x)\,dx
\]

Suppose we use \( \|f\|_1 = \int_a^b |f(x)|\,dx \) on \( V \) and the usual absolute value on \( \mathbb{R} \). We showed earlier that the linear function \( I(f) \) is actually continuous at \( f = 0 \). Now I claim \( I(f) \) is continuous on \( V \); i.e., continuous everywhere.

Why? Using properties of the integral on continuous functions that we proved last quarter,

\[
|I(f) - I(g)| = I(f(g)) \leq I(|f - g|) = \|f - g\|_1.
\]

This means that given \( \varepsilon > 0 \), we can take \( \delta = \varepsilon \) and then \( \|f - g\|_1 < \varepsilon \) implies \( |I(f) - I(g)| < \varepsilon \) (which is the \( \varepsilon\delta \) definition of continuity at \( g \) (or \( f \)). In fact, since \( \delta \) depends only on \( \varepsilon \) and not on \( f \) or \( g \), we have proved that the function \( I(f) \) is uniformly continuous - a concept we are about to define.

**Extra Credit.** Is \( I(f) \) still continuous when we replace the norm \( \|f\|_1 \) on \( V \) with \( \|f\|_2 \)? Explain your answer.

2) Look again at the function \( f(x, y) = \frac{x^2 - y^2}{x + y} \), for \( (x, y) \neq (0, 0) \) in \( \mathbb{R}^2 \). We know that this function cannot be continuous at \((0,0)\) since it has no limit as \((x,y) \to (0,0)\). There are more such examples in the homework.

**Definition 5** Suppose that \( V, W \) are normed vector spaces and \( U \) is a subset of \( V \). We say that \( f : U \to W \) is **uniformly continuous** at on \( U \) iff \( \forall \varepsilon > 0 \ \exists \delta > 0 \) (with \( \delta \) depending only on \( \varepsilon \)) such that \( \forall \ u, v \in U, \ \|u - v\| < \delta \) implies \( \|f(u) - f(v)\| < \varepsilon \).

The point here is that \( \delta \) does not depend on \( u, v \in U \).

Example 1 just considered is an example of a uniformly continuous function where in fact \( \delta = \varepsilon \). If I were a good person and lectured on the section of Lang about continuous functions on compact sets, we’d have many more examples of uniformly continuous functions, since Theorem 2.5 on page 198 of Lang says the following.

**Theorem 6** Suppose \( K \) is a compact subset of a normed vector space \( V \) and \( W \) is any normed vector space. A continuous function \( f : K \to W \) must be uniformly continuous on \( K \).

**What does "compact set" mean?** The definition can be found in Lang, p. 193. Let \( V \) be a finite dimensional normed vector space like \( \mathbb{R}^n \). Then \( K \subset \mathbb{R}^n \) is compact iff \( K \) is closed and bounded (This is a theorem not the definition of compact). So, for example, a closed ball of radius \( r \); i.e., \[ \{x \in \mathbb{R}^n \mid \|x-a\| \leq r \}\] is compact. This is false in infinite dimensions. A ball of radius \( r \) in infinite dimensions is not compact. Very inconvenient. Anyway this means that for \( f : \{x \in \mathbb{R}^n \mid \|x-a\| \leq r \} \to W \) continuity implies uniform continuity.

**More Examples.**

3) \( V=\text{normed vector space} \). Let \( f(x) = \|x\| \). Then \( f \) is uniformly continuous on \( V \). For all \( x, y \in V \), the triangle inequality implies (as it did for ordinary absolute value in an early homework problem from 142a),

\[
\|\|x\| - \|y\|\| \leq \|x - y\|.
\]
This says we can take $\varepsilon = \delta$ again.

4) Suppose that $L : \mathbb{R}^n \to \mathbb{R}^m$ is linear. Then take $e_j$ to be the standard basis vector $e_j = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}$, with a 1 in the jth row and the rest of the entries being 0. Every vector $v \in \mathbb{R}^n$, can be written uniquely in the form:

$$v = \begin{pmatrix} v_1 \\ \vdots \\ v_{j-1} \\ v_j \\ v_{j+1} \\ \vdots \\ v_n \end{pmatrix} = \sum_{j=1}^{n} v_j e_j.$$

It follows from linearity of $L$, that

$$Lv = \sum_{j=1}^{n} v_j Le_j.$$

Write $Le_j = \begin{pmatrix} a_{1j} \\ \vdots \\ a_{j-1,j} \\ a_{jj} \\ \vdots \\ a_{mj} \end{pmatrix} \in \mathbb{R}^m$. So we see that

$$Lv = \sum_{j=1}^{n} v_j \begin{pmatrix} a_{1j} \\ \vdots \\ a_{j-1,j} \\ a_{jj} \\ \vdots \\ a_{mj} \end{pmatrix} = Av,$$

where we multiply the matrix $A$ whose entries are $a_{ij}$ with the vector $v$. According to homework 2, problem 4, the linear function $L$ is uniformly continuous. You can see this by using the infinity norm on $\mathbb{R}^n$ and $\mathbb{R}^m$ and showing that there is a constant $C > 0$ so that $\|Lx\|_\infty \leq C \|x\|_\infty$. The constant $C$ depends on the entries $a_{ij}$ of the matrix $A$. If you take the $K = \max |a_{ij}|$, then $C = nK$ should work.

3 Completeness of $C[a,b]$ with respect to the $\infty$ Norm.

In part 2 of the Lectures we promised to prove the following.

**Theorem 7** The normed vector space $C[a,b]$ of continuous real valued functions on the finite interval $[a,b]$ is complete with respect to the norm $\|f\|_\infty = \max_{x \in [a,b]} |f(x)|$.

**Proof.** Recall that $V$ "complete" means every Cauchy sequence in $V$ converges to an element of $V$. So let $\{f_n\}$ be a Cauchy sequence in $C[a,b]$ using the norm $\|f\|_\infty$. This means for every $x \in [a,b]$, the sequence $\{f_n(x)\}$ of real numbers is Cauchy; as $\forall \varepsilon \exists N_\varepsilon$ s.t.

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty < \varepsilon \quad \text{when} \quad n, m \geq N_\varepsilon. \quad (1)$$
We showed last quarter that Cauchy sequences of real numbers converge to a limit in \(\mathbb{R}\). Thus \(\forall x \in [a, b]\) there is a function \(f(x) = \lim_{n \to \infty} f_n(x)\). Now we need to show that \(f_n\) converges uniformly to \(f\) on \([a, b]\).

Let \(\varepsilon > 0\) be given. There is \(M = M(x, \varepsilon) \geq N_\varepsilon\) so that \(m \geq M\) implies \(|f_m(x) - f(x)| < \varepsilon\). Then for \(n \geq N_\varepsilon\) we have the following sneaky formula by adding and subtracting \(f_m(x)\) and using the triangle inequality:

\[
|f(x) - f_n(x)| \leq |f(x) - f_m(x)| + |f_m(x) - f_n(x)| < \varepsilon + \|f_n - f_m\|_\infty < 2\varepsilon.
\]  

We chose \(N_\varepsilon\) so that formula (1) holds. This implies \(\|f - f_n\|_\infty < 2\varepsilon\) for \(n \geq N_\varepsilon\) which is uniform convergence of \(f_n\) to \(f\) on \([a, b]\) since \(N_\varepsilon\) does not depend on \(x\).

Next we need to show that \(f\) is continuous on \([a, b]\). To see this, note that for \(x, y \in [a, b]\), using the triangle inequality in a sneaky way again (this time adding and subtracting \(f_n(x) - f_n(y)\)):

\[
|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)|
\]

We know that for \(n \geq N_\varepsilon\) the 1st and 3rd terms are \(< 2\varepsilon\) by formula (2). Since \(f_n\) is continuous, there is a positive \(\delta\), depending on \(n, \varepsilon\) and \(y\) such that \(|x - y| < \delta\) implies the middle term is also \(< \varepsilon\). So the final result is that \(|f(x) - f(y)| < 5\varepsilon\). Replace \(\varepsilon\) by \(\varepsilon/5\), if you like. ■