Lecture 5. Power Series

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1 Power Series

Here we consider convergence and differentiation/integration of power series \( \sum_{n=0}^{\infty} a_n(x - p)^n \). I will mostly assume \( p = 0 \) for simplicity. We consider \( x \in \mathbb{R} \) mostly. The same arguments should work for complex numbers and matrices, with a little thought.

**Theorem.** a) Suppose \( \{a_n\} \) is a sequence of real numbers and \( \sum_{n=0}^{\infty} a_n x^n \) converges for some \( x \neq 0 \). Then \( \sum_{n=0}^{\infty} a_n u^n \) converges absolutely for all \( u \) such that \( |u| < |x| \).

b) If \( \sum_{n=0}^{\infty} a_n x^n \) diverges, then \( \sum_{n=0}^{\infty} a_n u^n \) diverges if \( |u| > |x| \).

**Proof.** a) \( \sum_{n=0}^{\infty} a_n x^n \) converges implies \( \lim_{n \to \infty} a_n x^n = 0 \). Since convergent implies bounded, we have \( |a_n x^n| \leq M \) for all \( n \). This implies if \( |u| < |x| \), we can apply the comparison test by writing

\[
|a_n u^n| = |a_n x^n u^n / x^n| \leq M \left| \frac{u^n}{x^n} \right| .
\]

So we can compare the series \( \sum_{n=0}^{\infty} |a_n u^n| \) with the convergent geometric series \( \sum_{n=0}^{\infty} \left| \frac{u^n}{x^n} \right| \).

We leave the proof of b) to the reader for extra credit. Hint: Use part a).

We are calling the following positive number the radius of convergence because we are really thinking of power series in the complex plane where we actually get an open circle of radius \( R \) where the power series converges. On the real line where this course lives, we only get an interval.

**Definition 1** Now define the **radius of convergence** \( R \) of \( \sum_{n=0}^{\infty} a_n x^n \)

\[
R = \text{l.u.b.} \left\{ r \mid r \geq 0, \sum_{n=0}^{\infty} |a_n| r^n \text{ converges} \right\} .
\]

I am using the convention here that \( \mathbb{R} \) (blackboard bold face font) is the set of real numbers and should not be confused with ordinary capital \( R \). This convention comes to us from Nicolas Bourbaki (the unreal mathematician).

**Corollary.** Let \( S \) denote the set of real numbers \( x \) such that \( \sum_{n=0}^{\infty} a_n x^n \) converges. Then \( S \) must be of the following forms: \( \{0\}, (-R, R), (-R, R], [-R, R), [-R, R], \) or \( (-\infty, \infty) = \mathbb{R} \).
Formulas for the Radius of Convergence $R$ of a Power Series $\sum_{n=0}^{\infty} a_n x^n$. We assume $a_n \neq 0$, $\forall n$.

1) Assume the limit $\lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right|$ exists. Then it is the radius of convergence $R$ for $\sum_{n=0}^{\infty} a_n x^n$. That is:

$$R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right|.$$

2) Again, assuming the limit involved exists, we have the formula:

$$\frac{1}{R} = \lim_{n \to \infty} |a_n|^\frac{1}{n}.$$

Proof. 1) Use the ratio test for absolute convergence of $\sum_{n=0}^{\infty} a_n x^n$. Setting $\lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = c$, we see that the ratio of the terms in the power series approaches a limit:

$$\left| \frac{a_{n+1} x^{n+1}}{a_n x^n} \right| = \left| \frac{a_{n+1}}{a_n} \right| \left| x \right| = \frac{\left| x \right|}{c}, \text{ as } n \to \infty.$$

The ratio test from Lectures Part 4 says the series converges if $|x| < c$, and diverges if $|x| > c$. Why? (Extra Credit). So $c = R$, the radius of convergence. This comes from the definition of radius of convergence as a least upper bound. If $|x| > R$, the series diverges and if $|x| < R$, the series converges. That is what we needed to show.

2) We leave this to the reader for Extra Credit.

Example 1). Our friend the geometric series $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$. The ratio of convergence is 1 by any of the formulas since $a_n = 1$, for all $n$.

Example 2). Our best friend the exponential series $\sum_{n=0}^{\infty} \frac{1}{n!} x^n = e^x$. Use the first formula for the radius of convergence.

Obtain

$$R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n \to \infty} \left| \frac{\frac{1}{n!}}{\frac{1}{(n+1)!}} \right| = \lim_{n \to \infty} \left| \frac{(n+1)!}{n!} \right| \lim_{n \to \infty} \left| \frac{(n+1)}{1} \right| = \infty.$$

This means the series converges everywhere.

The nice thing about this example is that the exponential series also converges for all complex numbers and for all nxn matrices. This gives a nice way to get the power series for sine and cosine out of Euler’s formula

$$e^{i\theta} = \cos \theta + i \sin \theta.$$

Example 3). Here’s a bad one: $\sum_{n=0}^{\infty} n! x^n$.

The radius of convergence is the reciprocal of that for Example 2. Thus you get $R = 0$. This series only converges at $x = 0$.

2 Differentiation and Integration of Power Series

We want to legally interchange $\int_a^b$ and $\lim$ (or $\sum_{n=0}^{\infty}$) so that we can integrate a power series term-by-term at least if we stay inside the radius of convergence. Similarly we want to legally interchange $\frac{d}{dx}$ and $\lim$ (or $\sum_{n=0}^{\infty}$).
Example.
Integrate the geometric series to get the power series for \( \log(1 - x) \). This is legal if \( |x| < 1 \) using the corollary of the following theorem.

\[
- \log(1 - x) = \int_0^x \frac{1}{1 - t} \, dt = \int_0^\infty t^n \, dt = \sum_{n=0}^{\infty} \int_0^x t^n \, dt = \sum_{n=0}^{\infty} \frac{t^{n+1}}{n+1} \bigg|_0^x = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}.
\]

Note that the formula makes sense if \( x = -1 \), although the following theorem and corollary do not justify the equality at \( -1 \).

**Theorem.** (Interchange of limit and integral). Suppose that \( f_n : [a, b] \to \mathbb{R} \) is continuous \( \forall n \) and converges uniformly on \([a, b]\) to \( f \). Then

\[
\lim_{n \to \infty} \int_a^b f_n = \int_a^b \lim_{n \to \infty} f_n = \int_a^b f.
\]

**Proof.** We already know that \( f \) must be continuous on \([a, b]\). So we can integrate it. See Lectures, Part 3, Thm. 7, p. 7. Using properties of the integral from last quarter, we find that

\[
\left| \int_a^b f - \int_a^b f_n \right| = \left| \int_a^b (f - f_n) \right| \leq \int_a^b |f - f_n| \leq \int_a^b \|f - f_n\|_{\infty} = \|f - f_n\|_{\infty} (b - a).
\]

It follows that given \( \varepsilon > 0 \), if we take \( N \) so that \( n \geq N \) implies \( \|f - f_n\|_{\infty} < \frac{\varepsilon}{b - a} \) (as we can by uniform convergence), then we have

\[
\left| \int_a^b f - \int_a^b f_n \right| < \varepsilon.
\]

**Corollary.** Suppose \( g_n(x) \) \([a, b] \to \mathbb{R}\) is continuous \( \forall n \) and \( \sum_{n=0}^{\infty} g_n(x) \) converges uniformly on \([a, b]\) to \( s(x) \). Then \( s(x) \) is continuous on \([a, b]\) and

\[
\sum_{n=0}^{\infty} \int_a^b g_n = \int_a^b \sum_{n=0}^{\infty} g_n.
\]

**Proof.** Extra Credit Exercise. ■

The proof of the last theorem was essentially one line. But differentiation requires more effort, as well as more hypotheses. Why should that be?

**Example.** Consider \( f_n(x) = \frac{\sin(nx)}{\sqrt{n}} \). This sequence converges uniformly to 0 on \([-\pi, \pi]\). Why? However,

\[
f_n'(x) = \sqrt{n} \cos(nx).
\]

This derivative has problems converging at all much less to 0! For example, it diverges at \( x = 0 \). In fact, it can be shown to diverge everywhere. Suppose \( x \neq 0 \). You just need to see that if \( x \neq 0 \), \( \exists N \) such that \( |\cos Nx| < \frac{1}{2} \) and thus

\[
|\cos(2Nx)| = |2 \cos^2 Nx - 1| = 1 - 2 \cos^2 N x > \frac{1}{2}.
\]

This means \( \exists n > N \) (namely \( n = 2N \)) such that \( |\cos(nx)| > \frac{1}{2} \). It follows that there is a subsequence satisfying \( |\sqrt{n_k} \cos(n_k x)| > \frac{1}{2} \sqrt{n_k} \) and thus diverging.

**Extra Credit.** Find an example to show that we need uniform convergence of \( f_n \) to \( f \) in the hypothesis of the preceding theorem. Pointwise convergence at each \( x \) in \([a, b]\) does not suffice for the interchange of integral and limit.

**Hint.** Look at \( f_n(x) = n^2 x(1 - x)^n \), on \([0, 1]\).
Theorem. (Interchange of Derivative and Limit). Suppose \( \{ f_n \} \) is a sequence of continuously differentiable functions on \([a, b]\). Suppose, also that the sequence of derivatives \( \{ f'_n \} \) converges uniformly to \( g \) on \([a, b]\). And, finally assume there exists one point \( x_0 \in [a, b] \) such that \( f_n(x_0) \) converges.

Then
\[
g(x) = \lim_{n \to \infty} \frac{d}{dx} f_n(x) = \frac{d}{dx} \lim_{n \to \infty} f_n(x).
\]
and the convergence of \( f_n(x) \) is uniform to the function \( f(x) \) with \( f'(x) = g(x) \), for all \( x \in [a, b] \).

\textbf{Proof.} By the fundamental theorem of calculus from last quarter, \( \forall n, \exists c_n \) s.t.
\[
f_n(x) = \int_a^x f_n'(c_n) \, dx.
\]
Here \( c_n = -f_n(a) \).

Let \( x = x_0 \) and take the limit of formula (1) as \( n \to \infty \). This says that \( \exists c = \lim_{n \to \infty} c_n \).

Next take the limit of formula (1) for general \( x \in [a, b] \) as \( n \to \infty \) using the preceding Theorem (which we can since the sequence of derivatives converges uniformly). This says, the sequence \( \{ f_n \} \) converges pointwise (i.e., for each fixed \( x \) in \([a, b]\)) to
\[
f(x) = \int_a^x g + c.
\]

To finish the proof of this theorem, we just need to see \( \{ f_n \} \) converges uniformly to \( f \). Well, try this, using our favorite triangle inequality and properties of integrals from last quarter,
\[
|f_n(x) - f(x)| = \left| \int_a^x f_n'(c_n) - \int_a^x g + c \right|
\]
\[
\leq \left| \int_a^x f_n' - \int_a^x g \right| + |c_n - c| \leq \int_a^x |f_n' - g| + |c_n - c|
\]
\[
\leq (b - a) \left\| f_n' - g \right\|_\infty + |c_n - c|.
\]

So if you insist on giving me an \( \varepsilon > 0 \), I can certainly find an \( N \) (depending only on \( \varepsilon \) and not on \( x \)) such that \( n > N \) implies \( |f_n(x) - f| < \varepsilon \). That is the meaning of uniform convergence. \( \blacksquare \)

\textbf{Corollary.} Suppose \( g_n(x) \mid [a, b] \to \mathbb{R} \) is continuously differentiable, \( \forall n \) and \( \sum_{n=0}^\infty g_n(x) \) converges uniformly on \([a, b]\) to \( s(x) \). And assume there is one point \( x_0 \in [a, b] \) such that \( \sum_{n=0}^\infty g_n(x) \) converges. Then we have \( \sum_{n=0}^\infty g_n(x) \) converges to \( r(x) \) uniformly on \([a, b]\) and \( s(x) = r'(x) \); i.e.,
\[
\sum_{n=0}^\infty \frac{d}{dx} g_n(x) = \frac{d}{dx} \sum_{n=0}^\infty g_n(x).
\]

\textbf{Proof. Extra Credit.} Deduc this corollary from the preceding theorem. \( \blacksquare \)

\textbf{Examples.} For our power series, these 2 theorems, allow us to integrate and differentiate term by term on closed subintervals of \((-R, R)\), where \( R \) is the radius of convergence. I find this somewhat surprising, since the power series you get by differentiation term-by-term blows up the term by a factor of \( n \):
\[
\frac{d}{dx} a_n x^n = n a_n x^{n-1}.
\]

It is also a bit surprising that integration does not produce a series that converges in a larger region since
\[
\int_0^z a_n t^n dt = a_n \frac{x^{n+1}}{n+1}, \quad \text{for} \quad n \geq 0.
\]

**Theorem.** Suppose \( f(x) = \sum_{n=0}^{\infty} a_n x^n \) with radius of convergence \( R \). Then for \( x \) in \((-R, R)\), we have the following facts.

1) \[
\int_0^z f(t) dt = \sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n+1}.
\]

2) \[
f'(x) = \sum_{n=0}^{\infty} na_n x^{n-1}.
\]

The integrated and differentiated series have the same radius of convergence \( R \).

**Proof.** Note that if \( R \) is its radius of convergence, a power series converges uniformly on closed subintervals of \((-R, R)\).

So we just have to convince ourselves that the extra factor of \( n \) in \( na_n \) does not affect the radius of convergence.

To see this, note that if \( 0 < |x| < c < R \), we know by the definition of radius of convergence \( R \) on the first page of this part of the lectures, that \( \sum_{n=0}^{\infty} |a_n| c^n < \infty \). Thus \( \lim_{n \to \infty} a_n c^n = 0 \). It follows that there is a bound \( M \) such that \( |a_n c^n| \leq M \), for all \( n \).

Therefore we have:

\[
n |a_n| |x|^{n-1} = \frac{n}{c} \left( \frac{|x|}{c} \right)^{n-1} |a_n| c^n \leq M \frac{n}{c} \left( \frac{|x|}{c} \right)^{n-1}.
\]

So \( \sum_{n=0}^{\infty} n |a_n| |x|^{n-1} \) can be compared with \( \sum_{n=0}^{\infty} n \left( \frac{|x|}{c} \right)^{n-1} \). This series converges by the ratio test as the ratios are

\[
\frac{(n+1) \left( \frac{|x|}{c} \right)^n}{n \left( \frac{|x|}{c} \right)^{n-1}} = \frac{n+1}{n} \frac{|x|}{c} \to \frac{|x|}{c} < 1, \quad \text{as} \quad n \to \infty.
\]

1) is left to the reader for Extra Credit. ■

**Examples.** The power series for \( e^x \), \( \sin(x) \), \( \cos(x) \) converge absolutely and uniformly on closed and bounded sets in the real line. So they are differentiable and integrable term by term.

## 3 Taylor Series

Last quarter we proved Taylor’s formula with remainder:

\[
f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k + R_n.
\]

We had 2 formulas for the remainder. The easiest to remember is \( R_n = \frac{f^{(k)}(c)}{k!} (x-a)^k \), for some \( c \) between \( a \) and \( x \). If one can show that \( \lim_{n \to \infty} R_n = 0 \), then the function \( f(x) \) is represented by its Taylor series within the radius of convergence. That is, for \( |x-a| < R \),

\[
f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k.
\]
See Wikipedia (Taylor series article) for animated pictures showing the convergence of the Taylor series to \(e^x\). We proved last quarter the our favorite functions: \(e^x\), \(\sin(x)\), \(\cos(x)\), \(\log(1-x)\) are represented by their Taylor series within the radius of convergence interval. So we have:

\[
e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n, \text{ for all } x \in R;
\]

\[
\log(1 - x) = -\sum_{n=0}^{\infty} \frac{1}{n} x^n, \text{ for } |x| < 1.
\]

Figure 1 shows the sum of the first 7 terms of the Taylor series for \(e^x\), namely \(f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!}\) in red and \(e^x\) in blue on the interval \([-5, 5]\).

Another example is the **Binomial Series**:

\[
(1 + x)^{\alpha} = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n, \tag{2}
\]

where we define the generalized **binomial coefficient**

\[
\binom{\alpha}{n} = \frac{\alpha(\alpha - 1) \cdots (\alpha - n + 1)}{n!}.
\]

As usual, define 0! = 1. When \(\alpha\) is a positive integer, this is the ordinary binomial theorem. Otherwise you need to restrict \(x\) to have \(|x| < 1\). Why?

**Extra Credit Exercise.** Prove formula (2) using Taylor’s formula from last quarter. Find the radius of convergence. Why is this the binomial theorem when \(\alpha\) is a positive integer?

We also saw last quarter that there are functions \(f(x)\) not represented by their Taylor series; e.g., \(f(0) = 0\), and for \(x \neq 0\), define \(f(x) = e^{-\frac{1}{x}}\). For this function, one can show that \(f^{(n)}(0) = 0\), for all \(n\). This means the Taylor series for \(f(x)\) around \(x = 0\) is 0 even though \(f(x)\) itself is positive except at \(x = 0\). For \(x \neq 0\),

\[
e^{-\frac{1}{x}} = f(x) \neq \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = 0.
\]

See Figure 2.
Figure 2: The plot of $f(x) = e^{-rac{1}{x^2}}$. This function is not represented by its Taylor series about 0 which is 0 for all $x$. 