Lecture 6. Construction of the Integral

Audrey Terras

May 18, 2010

1 Introduction

Recall Lang, Chapter 5. We assumed in Chapter 5 that for any finite interval \([a, b]\) there exists an integral \(\int_a^b f\) of a continuous function \(f : [a, b] \to \mathbb{R}\). And we assumed the integral satisfies the 2 axioms:

- **INT 1.** \(m \leq f(x) \leq M, \forall x \in [a, b] \implies m(b - a) \leq \int_a^b f \leq M(b - a)\).
- **INT 2.** \(a < c < b \implies \int_a^c f = \int_a^b f + \int_c^b f\).

We deduced all the other basic facts about integrals from these 2 axioms; e.g., the fundamental theorem of calculus, Taylor’s formula.

However, **we never showed that such an integral \(\int_a^b f\) exists!** I doubt that anyone was too worried. But now we will finally prove the existence of the integral. We will in fact be able to integrate more general functions than just those that are continuous on \([a, b]\).

How will we create \(\int_a^b f\)? It will be somewhat reminiscent of Math. 20B. Recall the Riemann sums, where you partition the interval \([a, b]\) with points \(a = a_0 < a_1 < \cdots < a_n = b\). Then you form rectangles the sum of whose areas approximates \(\int_a^b f\). As in Math. 20B by dividing up the \(x\)-axis interval \([0, 1]\) into 5 equal parts. The blue lines show the tops of the rectangles and also provide the graph of a step function approximating \(f(x) = x^2\). Our approximation for the integral is

\[
.2 \left( (.2)^2 + (.4)^2 + (.6)^2 + (.8)^2 + 1^2 \right) \approx .44
\]

The actual integral is \(\frac{x^3}{3}\bigg|_0^1 = 1/3 \approx .333333\).

The Math 142 (Lang) way of finding an approximation for the same integral \(\int_0^1 x^2 dx\) is illustrated in Homework 6 Problem 3 and Figure 2. We need to find a step function \(s(x)\) such that \(\|s - f\|_\infty = \sup_{0 \leq x \leq 1} |s(x) - f(x)|\) is small. To do this, one should 1st divide up the \(y\)-axis rather than the \(x\)-axis. Although in the end we divide up both axes.

For \(\int_0^1 x^2 dx\), we divide the \(y\)-axis interval (also \([0, 1]\)) into 5 equal subdivisions and go down to the \(x\)-axis via the inverse function \(\sqrt{x}\) to our original function \(x^2\). Now our approximation for the integral is:

\[
.2 \left( \sqrt{.2} - 0 \right) + .4 \left( \sqrt{.4} - \sqrt{.2} \right) + .6 \left( \sqrt{.6} - \sqrt{.4} \right) + .8 \left( \sqrt{.8} - \sqrt{.6} \right) + 1 \left( 1 - \sqrt{.8} \right) \approx .45026.
\]

Numerically this is not impressive. But it will give us a way of constructing integrals that throws a new light on the theory. We will be able to get the Lebesgue integral simply by changing our norm from \(\|\|_\infty\) norm to the \(\|\|_1\)-norm.

Our method of integrating \(x^2\) requires us to say the integral \(\int_0^1 x^2 dx\) is the limit as \(n \to \infty\) of \(\int_0^1 s_n(x) dx\) for a sequence of step function \(s_n(x)\) such that \(\lim_{n \to \infty} \|s_n(x) - x^2\|_\infty = 0\). More generally we write \(St[a, b]\)=the space of step functions on the interval \([a, b]\). We will be able to integrate functions \(f\) on \([a, b]\) that are uniform limits of sequences of step functions \(s_n(x)\). These functions are in \(St[a, b]\) = the closure (see definition below) of the space of step functions with respect to the \(\|\|_\infty\) norm. The official name for \(St[a, b]\) is the space of **regulated functions**. Yes, you can Google it.

In homework 6, problem 4, we give an example of a non-regulated function. This function can’t decide whether it is 1, -1 or 0 on any small interval containing 0. It does not have a right-hand limit at 0. The function \(\frac{x \sin(1/x)}{|x \sin(1/x)|}\) is pictured in figure 3. This is not quite the function in the homework problem as it does not have a value when \(x \sin(1/x) = 0\).

**Definition 1** If \(A\) is a subset of a normed vector space \(V\), define \(\overline{A}\) = the closure of \(A\) to be the set of points \(x \in V\) that are adherent to \(A\). Recall that \(x\) is adherent to \(A\) means that \(\forall \delta > 0, \exists a \in A\) s.t. \(\|x - a\| < \delta\).
Figure 1: The Math 20b way of approximating the integral \( \int_0^1 x^2 dx \). Divide the x-axis into 5 equal parts and take the value of the function at the right (or left) hand end point of the subinterval for the height of the rectangle.

Figure 2: The math 142 (Lang) way of approximating the integral \( \int_0^1 x^2 dx \). We find a step function in blue \( s(x) \) such that \( \| x^2 - s(x) \| \leq \frac{1}{5} \).
**Fact.** \( x \) is adherent to \( A \) \( \iff \) there is a sequence \( \{a_n\} \) of \( a_n \in A \) such that \( \lim_{n \to \infty} \|x - a_n\| = 0. \)

**Extra Credit.** Prove the Fact.

**Hint.** \( \implies \) Take \( \delta = 1/n. \)

\( \iff \) Use the definition of limit, with the \( \varepsilon \) in that definition equal to the \( \delta \) in the definition of adherent.
2 Continuous Linear Extension Theorem.

We will extend the integral from step functions to piecewise continuous functions and further using the continuous linear extension theorem. Suppose that $E$ and $F$ are normed vector spaces. Assume that $E$ is complete; i.e., all Cauchy sequences in $E$ converge to a limit in $E$. We will write $||| \cdot |||$ for both norms. In our case $E=\mathbb{R}$.

Suppose that $F_0$ is a subspace of $F$; i.e., a subset which is a vector space using the same operations of $+$ and multiplication by scalars as in $F$. In our case, $F_0$ will be $St[a,b]$, the step functions on the finite interval $[a,b]$.

Let $L: F_0 \to E$ be linear; i.e., $L(ax + by) = \alpha L(x) + \beta L(y)$, for all $x, y$ in $F_0$, and $\alpha, \beta \in \mathbb{R}$. In our case $L$ will be the integral over a finite interval $[a,b]$.

We also assume that $L$ is continuous for the norms on $E$ and $F_0$. We will use the sup (or least upper bound) norm on $St[a,b]$; i.e., $\|f\|_{\infty} = \sup_{x \in [a,b]} |f(x)|$.

**Lemma 1.** Suppose $L : F \to E$ is a linear function, where $E, F$ are normed vector spaces. Then $L$ is continuous $\iff \exists$ constant $C > 0$ such that $\|L(x)\| \leq C \|x\| \quad \forall x \in F$.

**Proof.** $\Rightarrow$ $L$ continuous implies it is continuous at $x = 0$. Thus, taking $\varepsilon = 1$, we see that $\exists \delta > 0$ such that $\|x\| < \delta$ implies $\|L(x)\| \leq 1$.

Now we use a trick coming from properties of the norm. If $x \neq 0$, (which we may assume without any problem since that case is pretty clear as $L(0) = 0$), we have $\left\| \frac{x}{2 \|x\|} \right\| = \frac{\delta}{2} < \delta$. So our constant $C = \frac{\delta}{2}$. It follows that $\|L(x)\| \leq \frac{\delta}{2} \|x\|$. So our constant $C = \frac{\delta}{2}$.

$\Leftarrow$ Now suppose $\|L(x)\| \leq C \|x\|$, $\forall x \in F$. Using the linearity of $L$, we see that $\|L(x) - L(y)\| = \|L(x - y)\| \leq C \|x - y\| < \varepsilon$, if $\|x - y\| < \frac{\varepsilon}{C} = \delta$. It follows that in fact $L$ is uniformly continuous on $F$. ■

**Lemma 2.** If $F_0$ is a subspace of a normed vector space $F$, then the closure $\overline{F_0}$ is also a subspace of $F$.

**Proof.** Recall (or see Lang, p. 153) that the closure $\overline{F_0}$ consist of points of $F$ which are limits of sequences of points from $F_0$. So we get this lemma from properties of limits. Let $x, y \in \overline{F_0}$, and $\{x_n\}$, $\{y_n\}$ be sequences of points from $F_0$ such that $x_n \to x$ and $y_n \to y$. Suppose $\alpha, \beta \in \mathbb{R}$. We need to show that $\lim_{n \to \infty} (\alpha x_n + \beta y_n) = \alpha x + \beta y$. This is an exercise (Extra Credit) in the properties of limits. It follows that $\alpha x + \beta y \in \overline{F_0}$. That means $\overline{F_0}$ is a vector space. ■

**Theorem.** Continuous Linear Extension Thm.

$F$-normed vector space. $F_0$ = subspace of $F$. $E$ = complete normed vector space. $L: F_0 \to E$, continuous linear can be extended to a unique continuous linear function $\overline{L} : \overline{F_0} \to E$. Here $\overline{F_0}$ is the closure of $F_0$, as in Lemma 2. If $C$ is the constant in Lemma 2 for $L$, then it also works for $\overline{L}$.

**Proof.** Suppose $x \in \overline{F_0}$. Then there is a sequence of points $\{x_n\}$ from $F_0$ such that $\lim_{n \to \infty} x_n = x$.

**Claim 1.** $\{L(x_n)\}$ is a Cauchy sequence in the complete normed vector space $E$.

**Proof of Claim 1.**

Using the linearity and continuity of $L$, we see that

$$\|L(x_n) - L(x_m)\| = \|L(x_n - x_m)\| \leq C \|x_n - x_m\| < \varepsilon$$

if $n, m$ are sufficiently large, since $\{x_n\}$ is Cauchy.

**Q.E.D. claim 1.**

Since $L(x_n) \in E$, a complete normed vector space, we know

$$\exists \lim_{n \to \infty} L(x_n) = w \in E.$$ 

We want to define

$$w = \overline{L}(x).$$

**Claim 2.** $w$ is unique and thus $w = \overline{L}(x)$ is a legal definition of a function.

**Proof of Claim 2.** Suppose we take another sequence $\{u_n\}$ from $F_0$ such that $\lim_{n \to \infty} u_n = x$. Then

$$\lim_{n \to \infty} L(u_n) = v \in E.$$
We want to show that $v = w$.

Well, we have

$$\|L(x_n) - L(u_n)\| \leq C \|x_n - u_n\| \to 0, \text{ as } n \to \infty.$$  

Using the triangle inequality, we see that

$$\|v - w\| \leq \|v - L(u_n)\| + \|L(u_n) - L(x_n)\| + \|L(x_n) - w\|$$

But each of the 3 terms on the right approaches 0 as $n \to \infty$. By the Squeeze Lemma then $\|v - w\| = 0$.

Q.E.D. claim 2.

Claim 3. The function $L(x)$ is linear.

Extra Credit Exercise using properties of limits and the linearity of $L$.

Claim 4. The function $L(x)$ is continuous with the same constant $C$ as for $L$.

Proof of Claim 4.

By the definition of $L(x)$ and the continuity of the norm, if $\lim_{n \to \infty} x_n = x$, for a sequence $\{x_n\}$ from $F_0$, we have

$$\|L(x)\| = \left\| \lim_{n \to \infty} L(x_n) \right\| = \lim_{n \to \infty} \|L(x_n)\|.$$  

We know that $\|L(x_n)\| \leq C \|x_n\| \ \forall n$. Since the limit preserves $\leq$, we see that

$$\|L(x)\| \leq \lim_{n \to \infty} C \|x_n\| = C \|x\|.$$  

Q.E.D. claim 4. □

This completes our proof of the continuous linear extension theorem - the hardest part of our construction of the integral. Next we proceed to use the theorem.