1 The Integral of a Step Map on $[a,b]$

We assume $-\infty < a < b < \infty$. A step map or function is just what it says; i.e., a function whose graph looks like a bunch of steps. More precisely, we define $f : [a, b] \rightarrow \mathbb{R}$ to be a step function if we can partition $[a, b]$ by $\mathcal{P} = \{a_0 < a_1 < \cdots < a_n\}$ so that $f$ is constant on each subinterval $(a_{i-1}, a_i)$, for each $i = 1, \ldots, n$:

$$f(x) = w_i, \; \forall x \in (a_{i-1}, a_i).$$

We don’t really care what values are assigned at the endpoints of the subintervals. See Figure 1.

![Figure 1: A step function $y = f(x)$.

Definition 1 The Riemann integral of the step function $f$ is defined to be

$$\int_a^b f = I(f, \mathcal{P}) = \sum_{i=1}^{n} w_i (a_i - a_{i-1}).$$

The sum $\sum_{i=1}^{n} w_i (a_i - a_{i-1})$ should look like a Riemann sum. For the step function pictured in Figure 1, the integral is just the sum of the areas of the 7 rectangles which make up the 7 steps. And $w_i = f(c_i)$ for any $c_i \in (a_{i-1}, a_i)$. 


We define a partition \( Q = \{ b_0 < b_1 < \cdots < b_m \} \) to be a refinement of the partition \( P = \{ a_0 < a_1 < \cdots < a_n \} \) if the set of points \( \{ a_0, a_1, \ldots, a_n \} \) is contained in the set of points \( \{ b_0, b_1, \ldots, b_m \} \).

So, for example, the points \( P = \{ \frac{i}{n} \mid i = 0, \ldots, n \} \) form a partition (regular) of \([0,1]\). The set \( Q = \{ \frac{i}{2n} \mid i = 0, \ldots, 2n \} \) is a refinement of \( P \).

Now we need to worry about the possibility that the integral of a step function might depend on what partition we choose.

**Lemma 1.** Suppose \( f \) is a step function with respect to 2 partitions \( P = \{ a_0 < a_1 < \cdots < a_n \} \) and \( Q = \{ b_0 < b_1 < \cdots < b_m \} \) of \([a,b]\). Then \( I(f,P) = I(f, Q) \).

**Proof. Step 1.** The partitions \( P \) and \( Q \) have a common refinement \( R \) whose subinterval endpoints are the union of the endpoints from the partitions \( P \) and \( Q \); i.e., the points \( \{ a_0, a_1, \ldots, a_n \} \cup \{ b_0, b_1, \ldots, b_m \} \). Call the common refinement \( R = \{ a = r_0 < r_1 < \cdots < r_k = b \} \).

**Step 2.** We claim that \( I(f,P) = I(f,R) = I(f, Q) \).

To see this, look at Figure 2.

![Figure 2: Inserting a partition point in the ith subinterval of \( P \) replaces a term in \( I(f,P) = \int_a^b f \) by the sum of 2 terms representing the sums of the areas of the 2 rectangles pictured (if the function is \( \geq 0 \) on the ith subinterval of \( P \) anyway) but this does not change the final result.](image)

This shows that if we look at any subinterval of partition \( P \), like \((a_{i-1}, a_i)\) and further subdivide it by inserting a point \( c \) in \( a_{i-1} < c < a_i \), then since \( f \) is the constant \( w_i \) on the subinterval \((a_{i-1}, a_i)\), the contribution to \( I(f,P) \) from \((a_{i-1}, a_i)\) is

\[ w_i(a_i - a_{i-1}) = w_i(c - a_{i-1}) + w_i(a_i - c). \]

The right hand side is the sum associated to the 2 subintervals \((a_{i-1}, c)\) and \((c, a_i)\) in our new partition refined by adding one point to the ith subinterval. Keep doing this for as many points as you want to add to \( P \) to get the refinement \( R \). Hopefully this convinces you that \( I(f,P) = I(f,R) \). Similarly, since \( R \) is also a refinement of \( Q \), we see that \( I(f,R) = I(f, Q) \). It follows that \( I(f,P) = I(f, Q) \).

**Lemma 2.** Suppose that \( St[a,b] \) denotes the set of all step functions defined on the interval \([a,b]\). Define the integral \( \int_a^b f \) for \( f \in St[a,b] \) as in Definition 1. By Lemma 1, the integral is independent of the partition \( P \) used to define the step function \( f \). Then the map \( I(f) = \int_a^b f \) is a continuous linear map from \( St[a,b] \) into \( \mathbb{R} \) using the \( \| \cdot \|_\infty \) norm on \( St[a,b] \) and
the usual absolute value on \( \mathbb{R} \). In particular,
\[
\left| \int_a^b f \right| \leq (b - a) \| f \|_\infty.
\]

**Proof. Continuity.** Note that if \( f \) is a step function with respect to the partition \( \mathcal{P} = \{ a_0 < a_1 < \cdots < a_n \} \) of \([a, b]\) and for each \( i = 1, \ldots, n \):
\[
f(x) = w_i, \ \forall x \in (a_{i-1}, a_i),
\]
then
\[
I(f, \mathcal{P}) = \int_a^b f = \sum_{i=1}^n w_i(a_i - a_{i-1}).
\]
We have \( \| f \|_\infty = \max_{i=1}^n |w_i| \). It follows, using the triangle inequality and the fact that the lengths of the subintervals of the partition add up to \( b - a \), that
\[
\left| \int_a^b f \right| \leq \sum_{i=1}^n |w_i| (a_i - a_{i-1}) \leq \sum_{i=1}^n \| f \|_\infty (a_i - a_{i-1})
\]
\[
= \| f \|_\infty \sum_{i=1}^n (a_i - a_{i-1}) = (b - a) \| f \|_\infty.
\]
Continuity at \( f = 0 \) follows from this inequality. Once we have proved linearity of the integral, the (uniform) continuity will also follow since then we can say
\[
\left| \int_a^b f - \int_a^b g \right| = \left| \int_a^b (f - g) \right| \leq \int_a^b |f - g| \leq (b - a) \| f - g \|_\infty.
\]
One quickly finds a \( \delta \) as a function of \( \varepsilon \) and independent of \( f \) and \( g \).

**Linearity.** Let \( \alpha, \beta \in \mathbb{R} \). Given step function \( f \) with corresponding partition \( \mathcal{P} \) and step function \( g \) with partition \( \mathcal{Q} \), we take a common refinement \( \mathcal{R} \) of \( \mathcal{P} \) and \( \mathcal{Q} \). Both \( f \) and \( g \) are step functions with respect to \( \mathcal{R} \). Let \( \mathcal{R} = \{ a_0 < a_1 < \cdots < a_n \} \). Suppose that for each \( i = 1, \ldots, n \):
\[
f(x) = w_i, \ \forall x \in (a_{i-1}, a_i)
\]
\[
g(x) = v_i, \ \forall x \in (a_{i-1}, a_i).
\]
Then \( \alpha f + \beta g \) is a step function for partition \( \mathcal{R} \) with values for each \( i = 1, \ldots, n \):
\[
(f + g)(x) = \alpha w_i + \beta v_i, \ \forall x \in (a_{i-1}, a_i).
\]
This shows that \( St[a, b] \) is indeed a normed vector space with norm \( \| \|_\infty \).

It follows from our definition of the integral on step functions that
\[
I(\alpha f + \beta g, \mathcal{R}) = \int_a^b (\alpha f + \beta g) = \sum_{i=1}^n (\alpha w_i + \beta v_i) (a_i - a_{i-1})
\]
\[
= \alpha \sum_{i=1}^n w_i(a_i - a_{i-1}) + \beta \sum_{i=1}^n v_i(a_i - a_{i-1})
\]
\[
= \alpha \int_a^b f + \beta \int_a^b g.
\]
So the integral is linear.
2 Properties of the Integral on Step Functions.

Recall that last quarter while doing Chapter 5 of Lang, p. 101, we saw that in order to prove the fundamental theorem of calculus, we wanted our integrals to have 2 basic properties of the integral on step functions \( f \in St[ab] \).

Int 1. If \( m \leq f(x) \leq M \) for all \( x \in [a, b] \), then \( m(b - a) \leq \int_a^b f \leq M(b - a) \).

Int 2. If \( a < c < b \), then \( \int_a^c f = \int_a^b f + \int_c^b f \).

Proof of Int 2. For \( f \) a step function on \([a, b] \). We can always add a point like \( c \) to our partition \( P = \{a_0 < a_1 < \cdots < a_n\} \) of \([a, b] \) defining \( f \) and assume \( c = a_j \) for some \( j \). Suppose that for each \( i = 1, \ldots, n \):

\[
f(x) = w_i, \quad \forall x \in (a_{i-1}, a_i).
\]

Then according to our definition of the integral of a step function

\[
I(f, P) = \int_a^b f = \sum_{i=1}^n w_i(a_i - a_{i-1})
= \sum_{i=1}^j w_i(a_i - a_{i-1}) + \sum_{i=j+1}^n w_i(a_i - a_{i-1})
= \int_a^c f + \int_c^b f.
\]

Proof of Int 1. \( m \leq f(x) \leq M \) for all \( x \in [a, b] \) means \( m \leq w_i \leq M \) for each \( i = 1, \ldots, n \). It follows since the sum of the lengths of the subintervals of \( P \) is \( b - a \),

\[
m(b - a) = \sum_{i=1}^n m(a_i - a_{i-1}) \leq \sum_{i=1}^n w_i(a_i - a_{i-1}) \leq \sum_{i=1}^n M(a_i - a_{i-1}) = M(b - a).
\]

This completes our proofs of the properties of the integral on step functions.

Remark 2 We could replace Int 1 by the statement that

Integrals Preserve

\[
f, g \in St[a, b] \text{ and } f(x) \leq g(x), \text{ for all } x \in [a, b], \text{ then } \int_a^b f \leq \int_a^b g.
\]

In the next section we prove a result which will help us in our scheme to integrate continuous functions.

3 Approximation of Continuous Functions by Step Functions

Theorem 3 \( f : [a, b] \to \mathbb{R} \) continuous implies \( f \) is uniformly continuous on \([a, b] \).

Proof. by Contradiction.

Otherwise \( \exists \varepsilon > 0 \text{ s.t. } \forall n \in \mathbb{Z}^+ \exists x_n, y_n \in [a, b] \) with \( |x_n - y_n| < \frac{1}{n} \) and \( |f(x_n) - f(y_n)| \geq \varepsilon \).

But we showed last quarter (using the Spanish Hotel argument in part 2 of the Lectures) that any bounded sequence of real numbers has a convergent subsequence. Thus there is an infinite subset \( J \) of \( \mathbb{Z}^+ \) such that

\[
\exists \lim_{n \to \infty} n \Rightarrow x_n = x \in [a, b]
\]

\[ n \in J \]
and
\[ \lim_{n \to \infty} y_n = y \in [a, b], \]

But then \( |x_n - y_n| < \frac{1}{n} \) implies using the continuity of absolute value:

\[
|x - y| = \left| \lim_{n \to \infty} x_n - \lim_{n \to \infty} y_n \right| \\
= \lim_{n \to \infty} |x_n - y_n| \leq \lim_{n \to \infty} \frac{1}{n} = 0,
\]

which means \( x = y \).

On the other hand, since \( |f(x_n) - f(y_n)| \geq \varepsilon \),

\[
0 = |f(x) - f(y)| = \left| \lim_{n \to \infty} f(x_n) - \lim_{n \to \infty} f(y_n) \right| \\
= \lim_{n \to \infty} |f(x_n) - f(y_n)| \geq \varepsilon.
\]

This contradiction proves the theorem. \( \blacksquare \)

Now we are ready to uniformly approximate continuous functions by step functions.

**Theorem 4** For every continuous function \( f : [a, b] \to \mathbb{R} \), there exists a sequence of step function \( s_n \in St[a, b] \) such that \( \|f - s_n\|_\infty \to 0 \) as \( n \to \infty \).

**Proof.** Given \( \varepsilon > 0 \), the preceding theorem says

\[
\exists \delta > 0 \text{ s.t. } |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon. \tag{1}
\]

Now let \( n \) be so large that \( \frac{b-a}{n} < \delta \) and let \( P = \{a_0 < a_1 < \cdots < a_n\} \) be a partition of \([a, b]\) such that \( a_i - a_{i-1} = \frac{b-a}{n} < \delta \). Define the step function \( s_n \) on the subinterval \((a_{i-1}, a_i)\) to take any value \( f(c) \) for \( c \in (a_{i-1}, a_i) \). Then by formula (1)

\[
|f(x) - s_n(x)| < \varepsilon \text{ for all } x \in (a_{i-1}, a_i).
\]

This means \( \|f - s_n\|_\infty < \varepsilon \) for \( n \) sufficiently large. \( \blacksquare \)

The preceding theorem says that the closure of the space of step functions with respect to the \( \|\|_\infty \) norm, written \( St[a, b] \), contains all continuous functions on \([a, b]\). It clearly also contains all piecewise continuous functions (which are allowed a finite number of discontinuities).

The space \( St[a, b] \) is called the space of **regulated functions**. We saw in the homework that not all bounded functions on \([a, b]\) are regulated. A function \( f(x) \) must have right and left hand limits at all \( x \in [a, b] \) to be regulated.
4 Extension of the Integral to the Regulated Functions on \([a,b]\).

We can use the continuous linear extension theorem to extend the integral from \(St[a,b]\) to \(\overline{St}[a,b]\). Everyone \(f \in \overline{St}[a,b]\) is a limit with respect to the \(\|\|_\infty\) norm of a sequence \(s_n \in St[a,b]\). Then the continuous linear extension theorem tells us to define

\[
\int_a^b f = \lim_{n \to \infty} \int_a^b s_n.
\]

We need to show that this extended integral has the desired properties (mainly Int 1 and Int 2 from Lang Chapter 5).

Theorem 5 Properties of the Integral on the Space \(\overline{St}[a,b]\) of Regulated Functions (which includes all continuous, even piecewise continuous functions on \([a,b]\)).

1) \(\int_a^b f\) is a linear map from \(f \in \overline{St}[a,b]\) into \(f \in \mathbb{R}\).

\[
\left| \int_a^b f \right| \leq (b-a) \|f\|_\infty.
\]

2) \(\int_a^b f\) preserves inequalities; i.e., \(f,g \in \overline{St}[a,b]\) with \(f(x) \leq g(x)\ \forall x \in [a,b]\) implies

\[
\int_a^b f \leq \int_a^b g.
\]

3) \(a < c < b\) implies \(\int_a^b f = \int_a^c f + \int_c^b f\).

Proof. 1) This is just the continuous linear extension theorem from the last lecture.

2) Let \(h(x) = g(x) - f(x)\ \forall x \in [a,b]\). Then \(h(x) \geq 0\ \forall x \in [a,b]\). Suppose \(s_n \in St[a,b]\) s.t. \(\lim_{n \to \infty} \|s_n - h\|_\infty = 0\).

Suppose \(s_n\) is a step function for the partition \(P = \{a_0 < a_1 < \cdots < a_m\}\) of \([a,b]\) and for \(i = 1,\ldots,m\),

\(f(x) = w_i, \ \forall x \in (a_{i-1}, a_i)\).

If \(w_i < 0\) for some \(i\), we can replace \(w_i\) by 0 and only make a new step function \(s_n^*\) which is even closer to \(h\) than \(s_n\) was (because \(f(x) \geq 0\) for all \(x\)). Now we see that

\[
\int_a^b g - \int_a^b f = \int_a^b h = \lim_{n \to \infty} \int_a^b s_n^* \geq 0.
\]

3) Let \(s_n\) be a sequence of step maps converging to \(f\) in the \(\infty\)-norm on \([a,b]\). Then \(s_n\) must also converge to \(f\) in the \(\infty\)-norm on \([a,c]\) and on \([c,b]\). We showed that for step functions we have:

\[
\int_a^b s_n = \int_a^c s_n + \int_c^b s_n.
\]

Take the limit as \(n \to \infty\) to get property 3 using the basic properties of limits from last quarter.
This completes our discussion of the existence of an integral with the properties Int1 and Int 2 on the first page of Lang’s Chapter 5. From this we deduced all the basic properties of integrals such as the fundamental theorem of calculus, integration by parts, the formula for substituting in an integral, the formulas like

$$\int_{a}^{b} x^n \, dx = \frac{x^{n+1}}{n+1} \bigg|_{x=a}^{b} = \frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1},$$

when $n \neq -1$ and assuming that if $n < -1$, 0 is not in the interval $[a, b]$.

To extend the fundamental theorem of calculus from continuous functions to piecewise continuous functions or regulated functions, requires a little more effort. Lang does this in Chapter 10 and produces theorems legalizing differentiation under the integral sign. I leave it to you to read these things.

We move on to Chapter 11.