1) Define and give an example:
   a) radius of convergence of \( \sum_{n=0}^{\infty} a_n x^n \)
   b) adherent point to a set \( A \) in a normed vector space \( V \)
   c) continuous linear map \( L: V \to W \), where \( V, W \) are vector spaces
   d) step function on the interval \([a,b]\)
   e) Riemann integral of a step function
   f) partition of an interval \([a,b]\)
   g) convergence of a series \( \sum v_n \) in a normed vector space

2) a) State and prove the integral test.
    b) State and prove the comparison test.

3) Which of the series below converge and why?
   a) \( \sum_{n=2}^{\infty} \frac{1}{n(n-1)} \)
   b) \( \sum_{n=2}^{\infty} \frac{(-1)^n}{\log(n)} \)
   c) \( \sum_{n=2}^{\infty} \frac{1}{\sqrt{n(n-1)}} \)
   d) \( \sum_{n=1}^{\infty} \frac{\sin(n)}{2n^2 - n} \)
   e) \( \sum_{n=1}^{\infty} \frac{n}{2^n} \)

4) True - False. Tell whether the following statements are true or false. Give a brief reason for your answer.
   a) \( \sum_{n=0}^{\infty} a_n x^n \) converges implies \( \sum_{n=0}^{\infty} a_n u^n \) converges for all \( u \) such that \( |u| < |x| \).
   b) \( \sum_{n=1}^{\infty} \frac{1}{n} (x-2)^n \) converges if \( |x| < 1 \).
   c) \( f_n \) continuous on \([a,b]\) and \( \lim_{n \to \infty} f_n(x) = f(x) \forall x \in [a,b] \Rightarrow \lim_{n \to \infty} \int_a^b f_n = \int_a^b f \).
   d) Define \( f(x) = 0 \) for \( x \) irrational and \( f(x) = 1 \) for \( x \) rational. Then \( f \in \mathcal{S}[a,b] \) = the space of regulated functions on \([a,b]\).
   e) Any bounded function on \([a,b]\) can be uniformly approximated by step functions and is thus integrable.
   f) Any piecewise continuous function on \([a,b]\) can be uniformly approximated by step functions and is thus integrable.
   g) If \( \lim_{n \to \infty} a_n = 0 \), then \( \sum_{n=1}^{\infty} a_n \) is convergent.

5) a) State and prove a formula for the radius of convergence of \( \sum_{n=0}^{\infty} a_n x^n \).
    b) Apply your formula to find the radius of convergence of \( \sum_{n=1}^{\infty} \frac{1}{n} x^n \).
    c) What function does this power series represent within the interval where it converges?
6) \( a \in \) the set of adherent points to \( A = \overline{A} \iff \exists \text{ a sequence } a_n \in A \text{ s.t. } a = \lim_{n \to \infty} a_n. \)

7) a) Define \( \|g\|_{\infty} = \text{l.u.b.} \{ |g(x)| : x \in [0,1] \} \), where l.u.b. = least upper bound. Consider the function \( f(x) = x^3. \) Find a step function \( s \) on \([0,1]\) such that \( \|f-s\|_{\infty} \leq .25. \)

b) Compute \( \int_0^1 s \)

8) a) State and prove the continuous linear extension theorem.

b) How did we use the continuous linear extension theorem to extend the integral from step functions to continuous functions?

9) Suppose that \( f \) is a step function on \([a,b]\) with respect to partition \( P, \) and \( g \) is a step function on \([a,b]\) with respect to partition \( Q. \) Show that if \( R \) is a refinement of \( P \) and \( Q, \) then both \( f \) and \( g \) are step functions with respect to \( R, \) and \( \text{I}(f+g,R) = \text{I}(f,R) + \text{I}(g,R). \)

10) Show that the map which sends step function \( f \) to \( \int_a^b f \) is continuous with respect to the \( \| \|_{\infty} \) norm on the vector space \( \text{St}[a,b] \) of step functions on \([a,b].\)

11) Show that any continuous function on a finite closed interval is uniformly continuous on that interval.

12) a) Define the integral \( \int_a^b f \) for any function \( f \) which is a uniform limit of a sequence \( s_n \) of step functions on \([a,b].\)

b) Explain why you know that this integral is a continuous linear function of \( f, \) where continuity is with respect to the \( \| \|_{\infty} \) norm on the space of bounded functions on \([a,b].\)

13) Explain why the integral has the following 2 properties on the space \( C[a,b] \) of continuous functions on the finite interval \([a,b].\)

a) the integral preserves \( \leq \)

b) for any \( c \) with \( a < c < b, \) we have \( \int_a^b f = \int_a^c f + \int_c^b f. \)

14) Fundamental Theorem of Calculus. Suppose \( f: [a,b] \to \mathbb{R} \) is continuous. Show that then

\[
F(x) - F(a) = \int_a^x f \text{ for every } x \in [a,b] \iff F'(x) = f(x), \ \forall x \in [a,b].
\]

15) Let \( a < b < c. \) Suppose that \( f(x) \) is continuous on \([a,b]\) and we define \( g(x) \) to be equal to \( f(x) \) for all \( x \) in \([a,c]\) and for all \( x \) in \((c,b]\) but we define \( g(c) = f(c) + 100. \) Show that \( \int_a^b f = \int_a^b g. \)