MATH 20D – CORRECTED TAKE HOME FINAL – AUDREY TERRAS

Hand it in to the homework drop box for your TA before Thursday June 12, 2:30 p.m.

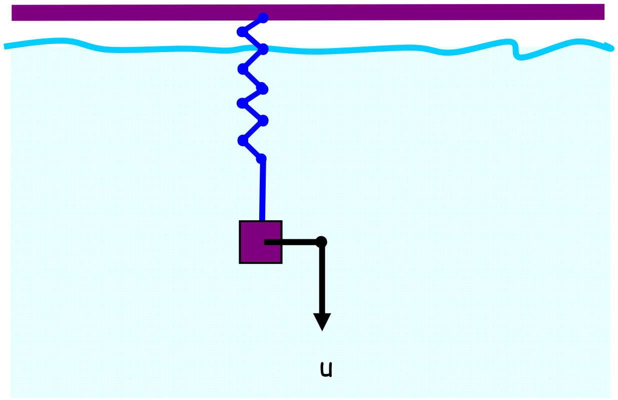
No late papers will be accepted!

The rules: You can work with other people and ask questions of your TAs & Audrey, and look at whatever books, websites (etc.) you want. But you MUST write on your exam the names of those you worked with and what books, etc. you used. And you MUST write up the answers to the exam in your own words. Papers with identical answers or word - for - word quotations from books will not be viewed as worth much. Please note that it is legal (even perhaps necessary) to use Matlab to draw the pictures from the text problems and the chaos problems below and to compute the eigenvalues, eigenvectors in problem 5.

The test concerns two applications of ODEs - the first to resonance and the second to chaos.

Part I. Resonance.

Read Section 3.8 and 3.9 of Boyce and DePrima. These sections involve the motion of a system such as that given by a small object of mass m attached to a spring of length L with Hooke's law constant k. The spring hangs down into a viscous medium with damping constant γ. The displacement of the spring from equilibrium is u(t).



Using Newton's law of motion one finds that u satisfies the ODE

$$mu''(t)+\gamma u'(t)+ku(t)=F(t)$$

where F denotes an upward force on the mass. This sort of ODE is also seen to model electric circuits on p. 202 of our text.

Resonance occurs when the spring is forced to vibrate at certain frequencies. This is described somewhat in Section 3.9 of Boyce and DePrima. One assumes that the forcing term on the right hand side of the ODE is $F(t)=a\cos(ft)$.

Problem 1. Do problem 7 on page 203 of Boyce and DePrima.

Problem 2. Do problem 14 on page 203 of Boyce and DePrima.

Problem 3. Do problem 19 on page 216 of Boyce and DePrima.

Problem 4. Do problem 20 on page 216 of Boyce and DePrima.

Problem 5. Do problem 31 on page 413 of Boyce and DePrima. It suffices to plot y_1 and y_2 versus t in parts d), e), f). Anyone who manages to make Matlab create a movie of the 2 springs vibrating gets humungous extra credit on this problem.

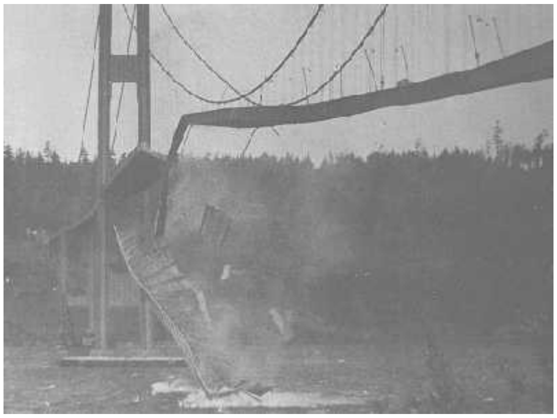
Problem 6. a) In problem 3 (3.9.19) of the text, we considered the ODE u″+u=3cos(wt), w not +1 or -1. Write down the solution u(t) and find the limit as w approaches 1. **You should see the spring blow up as time goes to infinity !**
You will need to use L'hopital's rule.

$$If \quad \lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0 \ \ (or \ \ \infty), \ \ then$$
$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

Here we assume the limit on the right exists. See Wikipedia for more info.

b) Write down the matrix equation y'=Ay+g corresponding to u″+u=3cos(wt), as we did in chapter 7. What does the resonance at w=1 and w=-1 have to do with the eigenvalues of the matrix A?

The blowing up of the spring in probem 6 when forced to vibrate at certain frequencies is a phenomenon called "**resonance**." One sees unbounded oscillations. In "real" life this would cause the spring to self destruct. Such things can happen in bridges or other structures. Most texts say the Tacoma Narrows bridge disaster was an example of such resonance. However, the latest research questions this conclusion. The Millenium pedestrian bridge in London showed such resonance and was closed for a time and fixed up to keep it from resonating with the many pedestrians walking across. Solders are required not to walk in step across bridges for similar reasons.



The original Tacoma Narrows Bridge opened to traffic on July 1, 1940. It collapsed just four months later during a 42-mile-per-hour wind storm on Nov. 7. There are many web sites devoted to the subject. See http://www.ketchum.org/bridgecollapse.html Movies of the collapse also exist on the web.

The **London Millennium Footbridge** is a pedestrian-only steel suspension bridge crossing the River Thames in London, England. The bridge opened on June 10, 2000 but unexpected lateral vibration (resonant structural response) caused the bridge to be closed on June 12 for modifications. The movements were produced by the sheer numbers of pedestrians (90,000 users in the first day, with up to 2,000 on the bridge at any one time).

M. Braun, *Differential Equations and their Applications*, notes: "There were many humorous and ironic incidents associated with the collapse of the Tacoma Bridge. When the bridge began heaving violently, the authorities notified Professor F. B. Farquharson of the University of Washington. Professor Farquharson had conducted numerous tests on a simulated model of the bridge and had assured everyone of its stability. The professor was the last man on the bridge. Even when the span was tilting more than twenty-eight feet up and down, he was making scientific observations with little or no anticipation of the imminent collapse of the bridge. "

"A large sign near the bridge approach advertised a local bank with the slogan 'as safe as the Tacoma Bridge.'"

"After the collapse of the Tacoma Bridge, the governor of the state of Washington made an emotional speech in which he declared 'We are going to build the exact same bridge, exactly as before.' Upon hearing this, the noted engineer Von Karman sent a telegram to the governor stating 'If you build the exact same bridge exactly as before, it will fall into the exact same river exactly as before.'"

There are also useful applications of resonance. Stuff in a microwave is forced to oscillate at the resonant frequency of water. Spectroscopy is another example of an application of resonance.

Part II. Chaos

The study of chaotic systems is rather recent. It is perhaps surprising that randomlike motions can come from completely deterministic systems.

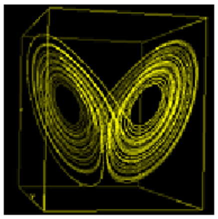
There is a Matlab demo showing the Lorenz attractor. In 1963 Edward Lorenz of M.I.T. published a paper containing a system of ODEs modelling convection in the atmosphere. The **Lorenz system differential equations** is:

$$x'=-sx+sy$$
$$y'=-xz+rx-y$$
$$z'=xy-bz$$

with s,r,b positive constants.

As the Matlab demo proceeds, you will see a point moving around in a weird orbit in 3-D space known as a strange attractor. The orbit is bounded, but not periodic and not convergent (hence the word "strange"). The most striking aspect of the solutions is that if you change the initial conditions only slightly, the solutions will be widely separated after a time.

The picture to the right is from *The Chaos Hypertextbook*, by Glenn Elert, which can be found on the web at http://hypertextbook.com/chaos/21.shtml



Lorenz realized that tiny changes in initial conditions would ultimately have a big effect on the solution after some time. So he coined the famous phrase "butterfly effect" that the mathematician in the movie Jurassic Park mentions. Lorenz asks: "Does the flap of a butterfly's wing in Brazil set off a tornado in Texas?" This quote begins the mystery *Death Qualified* by Kate Wilhelm.

Lorenz also said: "The average person, seeing that we can predict the tides pretty well a few months ahead would say, why can't we do the same thing with the atmosphere? It's just a different system, the laws are about as complicated. But I realized that any physical system that behaved nonperiodically would be *unpredictable*." These quotes were taken from Ian Stewart's book, *Does God Play Dice? The Mathematics of Chaos.*

A side note: Lorenz died recently. You can find his obituary from the New York times on the web. It gives the story of his discovery of the butterfly effect.

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Now let's look at a lower dimensional problem that has some of the qualities of the Lorenz attractor. This is the forced negative resistance oscillator found by Ueda and Akamatsu in 1981. We found it in the book *Chaos*, edited by Arun V. Holden.

The **system of differential equations for the forced negative resistance oscillator is**

$$x'(t)=y$$
$$y'(t)=(1-x^2)y-x^3+b\cos(ft)$$

We can use Matlab to solve this numerically as in Matlab Assignment #6.

The case b=1, f=1.617

First write the m-file chaos.m

```
        function dz=chaos(t,z)
        dz=[z(2);(1-z(1)^2)*z(2)-z(1)^3+cos(1.617*t)];
```

Then the m-file ueda.m is below. In this file [0,100] is the time interval and [1,.2] gives the initial conditions x(0)=1, y(0)=.2.

```
  % ueda forced resistance oscillator
        [t,z]=ode45('chaos',[0,100],[1,.2]);
        figure
        plot(z(:,1),z(:,2))
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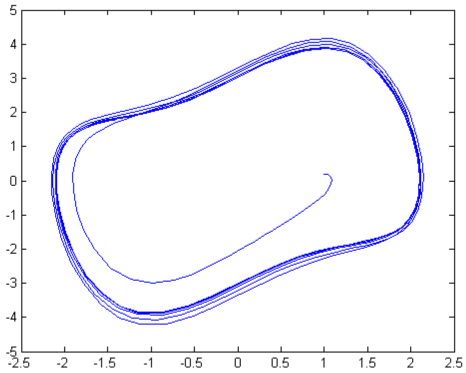
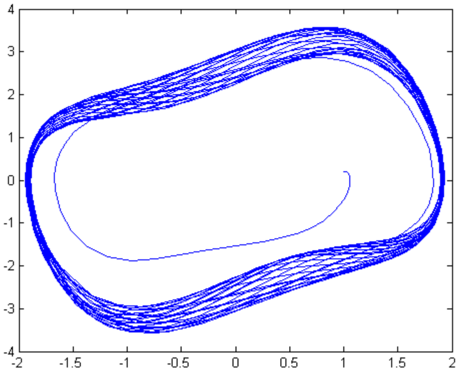
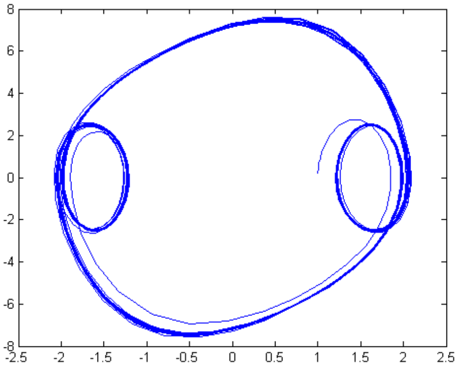


The 1st figure to the right plots the solution for b=1, f=1.617 to $x'(t)=y$, and $y'(t)=(1-x^2)y-x^3+b\cos(ft)$.



With b=1 and f=4, in $x'(t)=y$, and $y'(t)=(1-x^2)y-x^3+b\cos(ft)$, you get the 2nd figure.



With b=17, f=5 in $x'(t)=y$, and $y'(t)=(1-x^2)y-x^3+b\cos(ft)$, you get the 3rd figure.

Problem 7. Do more experiments on a system like **forced negative resistance oscillator**

$$x'(t)=y$$
$$y'(t)=(1-x^2)y-x^3+b\cos(ft)$$

- trying more values of the constants b and f - or even a different equation for y'(t) such as $y'(t)= -by-x^3+c\sin(ft)$. Put the figures in a Word Document.

Problem 8. Do some Matlab experiments with the **forced Brusselator**. This is a system of ODEs coming from chemistry. It was introduced by Prigogine and Lefever in 1968. Tomita and Kai added a forcing term in 1978. The system of ODEs is

$$x'(t)=A+x^2y-Bx-x+a\cos(ft)$$
$$y'(t)=Bx-x^2y$$

Try A=0.4, B=1.2, a=0.05. Let f=0.6, 0.8, 0.83, 0.84, 0.95
Initial conditions that might work are: x(0)=0.8, y(0)=0.6.

Problem 9. Do some Matlab experiments with **the glycolytic oscillator.** This is a system of ODEs coming from biology. The system of ODEs is

$$x'(t)=a+b\cos(ft)-xy^2$$
$$y'(t)=xy^2 - y$$

Try a=0.999, b=0.42, f=4.5, 3.5, 2.0, 1.8, 1.75.
Initial conditions that might work are: x(0)=1, y(0)=1.

Problem 10. Do the solutions to problems 7, 8, 9 illustrate the butterfly effect? That is, do tiny errors build up over time into huge errors so that predictions become nonsense? Do slightly different initial conditions ultimately lead to widely separated solutions?

Does this mean that Matlab might actually be lying to us? For we always approximate in these numerical methods. What do our pictures mean if huge errors can build up?

Ian Stewart, *Does God Play Dice?* (pages 287-288) says: "On several occasions we've run into a curious feature of chaotic dynamics: the same problem, run on different makes of computer, leads to different answers." Thus two computers predicting the weather will often come up with two different predictions.

"Despite this, if a hundred people draw Lorenz attractors on a hundred different makes of computer, they all see much the same shape."

"If you think you're solving the initial value problem for the Lorenz equations, with the exact numerical conditions that you fed into your computer, then you're fooling yourself. But if you think you're plotting out the shape of the attractor, rather than a trajectory of it, you're in good shape. Tiny errors that move your point away from the attractor rapidly die out - that's what 'attractor' means. It's only errors that stay on the attractor that blow up."