Fundamental Matrices, Matrix Exp & Repeated Eigenvalues – Sections 7.7 & 7.8

Given fundamental solutions

$$\vec{x}_1, \ldots, \vec{x}_n \quad of \quad the \quad ODE \quad \frac{d\vec{x}}{dt} = A\vec{x}$$

we put them in an nxn matrix

, $$\Psi(t) = \left(\vec{x}_1, \ldots, \vec{x}_n\right)$$

with each of the solution vectors being a column.

We call Ψ(t) a **fundamental matrix** for the system of ODEs.

Example.

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \vec{x}$$

$$\det \begin{pmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 - 4$$

$$= \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1)$$

So the eigenvalues of the matrix $A=\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

in our ODE are $\lambda=3,-1$.

The corresponding eigenvectors are found by solving $(A-\lambda I)\mathbf{v}=0$ using Gaussian elimination.

We find that

the eigenvector for eigenvalue 3 is: $v=\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

the eigenvector for eigenvalue -1 is: $w=\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

So the corresponding solution vectors for our ODE system are $\vec{u}_1 = e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \vec{u}_2 = e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Our fundamental matrix is:

$$\Psi(t) = \begin{pmatrix} \vec{u}_1 & \vec{u}_2 \end{pmatrix} = \begin{pmatrix} e^{3t} & e^{-t} \\ e^{3t} & -e^{-t} \end{pmatrix}$$

The **general solution** is

$$\vec{y}(t) = c_1\vec{u}_1 + c_2\vec{u}_2 = \Psi(t)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

If you need to find c_1, c_2 satisfying some initial condition like **$y(t_0)=b$**, then you need to solve

$$\vec{y}(t_0) = \Psi(t_0)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \Psi(t_0)^{-1}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

The inverse of the fundamental matrix exists because the Wronskian is not 0.

Matrix Exponential

Recall that the Taylor series for exp(x) is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad n! = n(n-1)\cdots 2\cdot 1.$$

The series converges for **all** real numbers x.

It also converges for all complex numbers x and for all nxn matrices A.

$$\exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!} = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

So exp(0)=I=identity matrix.

You can legally differentiate these Taylor series term by term. This says for scalar x and nxn matrix A, we have:

$$\frac{d\exp(xA)}{dx} = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d\left(xA\right)^n}{dx} = \sum_{n=0}^{\infty} \frac{n}{n!} x^{n-1} A^n$$

$$= A\sum_{n=1}^{\infty} \frac{1}{\left(n-1\right)!} \left(xA\right)^{n-1} = A\exp(xA)$$

And exp(0)=I, the identity matrix.

So we have a solution to the system of ODEs given by

$$\frac{d\vec{y}}{dx} = A\vec{y}, \quad \vec{y}(0) = \vec{b};$$

$$\vec{y} = \exp(xA)\vec{b}$$

When a matrix is diagonal D, it is easy to compute exp(D).

$$\exp\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix} + \frac{1}{3!}\begin{pmatrix} a^3 & 0 \\ 0 & b^3 \end{pmatrix} + \cdots$$

$$= \begin{pmatrix} 1 + a + \frac{a^2}{2!} + \cdots & 0 \\ 0 & 1 + b + \frac{b^2}{2!} \end{pmatrix} = \begin{pmatrix} e^a & 0 \\ 0 & e^b \end{pmatrix}$$

When an nxn matrix A has n linearly independent eigenvectors $\vec{v}_1, \ldots, \vec{v}_n$, corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_n$ we can write

$$A = TDT^{-1},$$

$$T = \left(\vec{v}_1 \ldots \vec{v}_n\right) \quad \& \quad D = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}$$

The reason is $A\vec{v}_j = \lambda_j \vec{v}_j$.

One has $$\exp(TDT^{-1}) = T\exp(D)T^{-1}$$

From this, it is easy to compute the solutions to the system of ODEs

$$\frac{d\vec{y}}{dx} = A\vec{y}, \quad \vec{y}(0) = \vec{b};$$

$$\vec{y} = \exp(xA)\vec{b}$$

$$A = TDT^{-1},$$

$$T = \left(\vec{v}_1 \ldots \vec{v}_n\right) \quad \& \quad D = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}$$

$Q(x)=\exp(xD)$ and the fundamental matrix is $\Psi(x)=TQ(x)$, where D is the diagonal matrix of eigenvalues of A and T is the matrix coming from the corresponding eigenvectors in the same order.

$\exp(xA)$ is a fundamental matrix for our ODE

Repeated Eigenvalues

When an nxn matrix A has repeated eigenvalues it may not have n linearly independent eigenvectors. In that case it won't be diagonalizable and it is said to be deficient.

Example.

$$A=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$$

$$\det\begin{pmatrix}1-\lambda & 1\\ 0 & 1-\lambda\end{pmatrix}=0$$

The roots of this are both 1.

Gaussian elimination solves (A-I)x=0

$$A-I=\begin{pmatrix}1-1 & 1\\ 0 & 1-1\end{pmatrix}=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$$

Solutions have x_2=0, x_1 arbitrary. So we have only 1 linearly independent eigenvector

$$\vec{v}=\begin{pmatrix}v_1\\ v_2\end{pmatrix}=\begin{pmatrix}1\\ 0\end{pmatrix}$$

This gives us one solution to the ode

$$\frac{d\vec{y}}{dx} = A\vec{y}; \qquad \vec{y} = e^x \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^x \\ 0 \end{pmatrix}$$

How to find a 2nd linearly independent solution?

Our first idea is just to multiply this by x, but that will not be linearly independent as it is a multiple of the eigenvector v.

Matrix Exp solves the problem.

AB=BA implies exp(A+B)=expAexpB

So our fundamental matrix is exp(xA)

A=I+N, N= $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$N^2=0$ implies exp(xN)=I+xN= $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$

Ψ(t)=exp(x(I+N))=exp(xI)exp(xN)

= $\begin{pmatrix} e^x & xe^x \\ 0 & e^x \end{pmatrix}$

Another way: write $\vec{u}_2 = xe^x\vec{v} + e^x\vec{w}$

for some constant vector **w** to be determined.

$$\frac{d\vec{w}}{dx} = \frac{d\left(xe^x\right)}{dx}\vec{v} + \frac{de^x}{dx}\vec{w}$$

$$= \left(e^x + xe^x\right)\vec{v} + e^x\vec{w}$$

$$\frac{d\vec{w}}{dx} = \left(e^x + xe^x\right)\vec{v} + e^x\vec{w}$$

$$= A(xe^x\vec{v} + e^x\vec{w})$$

$$= xe^x\vec{v} + e^x A\vec{w}$$

Here we used the fact that Av=v.

So now equate coefficients of x and 1

$$\left(e^x + xe^x\right)\vec{v} + e^x\vec{w} = xe^x\vec{v} + e^x A\vec{w}$$

$$\Leftrightarrow \begin{cases} \left(\vec{v} + \vec{w}\right) = A\vec{w} \\ x\vec{v} = x\vec{v} \end{cases}$$

We need to solve for w

$$\vec{v} = \left(A - I\right)\vec{w};$$

$$Gauss \quad (A - I \mid v) = \left(\begin{array}{cc|c} 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right)$$

Solution is

$x_2=1$, x_1 arbitrary.

Putting this all together:

$$\vec{u}_2 = xe^x\vec{v} + e^x\vec{w} = xe^x\begin{pmatrix}1\\0\end{pmatrix} + e^x\begin{pmatrix}0\\1\end{pmatrix}$$

$$\vec{u}_2 = \begin{pmatrix}(x+1)e^x \\ e^x\end{pmatrix}$$

Check that this gives the same fundamental matrix as exp did.