a stroll through
the zeta garden

Lecture 1: Riemann, Dedekind, Selberg, and Ihara Zetas

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more details can be
found in

my webpage:
www.math.ucsd.edu
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newbook.pdf

First the Riemann
Zeta
The Riemann zeta function for Re(s) > 1

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - p^{-s}\right)^{-1}. \]

- **Riemann (1859)** extended to all complex s with pole at s=1
- **Functional equation** relates value at s and 1-s
  \[ \Lambda(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s) = \Lambda(1-s) \]
- **Riemann hypothesis** (non real zeros ζ(s)=0 are on the line Re(s)=1/2). This now checked for \(10^{13}\) billion zeros. (work of X. Gourdon and P. Demichel). See Ed Pegg Jr.'s website.

Graph of \( z = |\zeta(x+iy)| \) showing the pole at x+iy=1 and the first 6 zeros which are on the line x=1/2, of course. The picture was made by D. Asimov and S. Wagon to accompany their article on the evidence for the Riemann hypothesis as of 1986.
duality between primes & complex zeros of zeta using Hadamard product over zeros

prime number theorem proved by Hadamard and de la Vallée Poussin (1896-1900). Their proof requires complex analysis

$$\# \{p = \text{prime} \mid p \leq x\} \sim \frac{x}{\log x}, \text{ as } x \to \infty$$

statistics of Riemann zero spacings studied by Odlyzko (GUE)


Odlyzko’s Comparison of Spacings of 7.8 × 10^7 Zeros of Zeta at heights ≈ 10^{20} & Eigenvalues of Random Hermitian Matrix (GUE).
Many Kinds of Zeta

Dedekind zeta of an algebraic number field $F$ such as $\mathbb{Q}(\sqrt{2})$, where primes become prime ideals $p$ and infinite product of terms

$$(1-Np^{-s})^{-1}, \text{ where } Np = \text{norm of } p = \#(O/p), O=\text{ring of integers in } F$$

**Functional Equations:** $\zeta_F(s)$ related to $\zeta_F(1-s)$

- **Hecke**

**Values at 0:**

$\zeta(0) = -1/2, \quad \zeta_F(0) = -hR/w$

$h = \text{class number}$ (measures how far $O_K$ is from having unique factorization) ($=1$ for $K=\mathbb{Q}(\sqrt{2})$)

$R = \text{regulator}$ (determinant of logs of units)

$= \log(1+\sqrt{2})$ when $K=\mathbb{Q}(\sqrt{2})$

$w = \text{number of roots of unity in } K$ is 2, when $K=\mathbb{Q}(\sqrt{2})$

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Statistics of Prime Ideals and Zeros

- From information on zeros of $\zeta_F(s)$ obtain **prime ideal theorem**

$$\#\{p \text{ prime ideal in } O_K \mid Np \leq x\} \sim \frac{x}{\log x}, \text{ as } x \to \infty$$

- There are an infinite number of primes such that $\left(\frac{2}{p}\right) = 1$.

- Dirichlet theorem: there are an infinite number of primes $p$ in the progression $a, a+d, a+2d, a+3d, \ldots$, when $\gcd(a,d)=1$.

- **Riemann hypothesis still open:**

  GRH or ERH: $\zeta_F(s)=0$ implies $\Re(s)=1/2$, assuming $s$ is not real.
**Selberg zeta** associated to a compact Riemannian manifold $M=\Gamma\backslash H$, $H =$ upper half plane with $ds^2=(dx^2+dy^2)y^{-2}$
\(\Gamma\) = discrete subgroup of group of real fractional linear transformations
primes = primitive closed geodesics $C$ in $M$ of length $\nu(C)$, (primitive means only go around once)

\[ Z(s) = \prod_{[C]} \prod_{j \geq 0} \left( 1 - e^{-(s+j)\nu(C)} \right) \]

Duality between spectrum $\Delta$ on $M$ & lengths closed geodesics in $M$
$Z(s+1)/Z(s)$ is more like Riemann zeta

Realize $M$ as quotient of upper half plane $H=\{x+iy | x,y \in \mathbb{R}, \ y>0\}$.
Non-Euclidean distance: $ds^2= y^{-2}(dx^2+dy^2)$
$ds$ is invariant under $z \rightarrow (az+b)/(cz+d)$, for $a,b,c,d$ real and $ad-bc=1$. PSL($2, \mathbb{R}$).

Corresponding Laplacian
\[ \Delta = y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \]

also commutes with action of PSL($2, \mathbb{R}$).

The curves (geodesics) minimizing arc length are circles and lines in $H$ orthogonal to real axis. Non-Euclidean geometry.
View compact or finite volume manifold as $\Gamma \backslash \mathcal{H}$, where $\Gamma$ is a discrete subgroup of $\text{PSL}(2,\mathbb{R})$. For example, $\Gamma = \text{PSL}(2,\mathbb{Z})$, the modular group. Fundamental Domain is a non-Euclidean triangle.
A closed geodesic in $\Gamma \backslash \mathbb{H}$ comes from one in $\mathbb{H}$. One can show that the endpoints in $\mathbb{R}$ of such a geodesic are fixed by hyperbolic elements of $\Gamma$; i.e., those $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with trace $= a+d>2$.

Primitive closed geodesics are traversed only once. They correspond to hyperbolic elements that generate their centralizer in $\Gamma$.

Next a picture of images of points on 2 geodesics circles after mapping them into a fundamental domain of $\text{PSL}(2,\mathbb{Z})$
We will see they have similar properties and applications to those of number theory. But first we need to figure out what primes in graphs are. This requires us to label the edges.

**Labeling Edges of Graphs**

\( X \) = finite connected (not-necessarily regular graph). Usually we assume: graph is not a cycle or a cycle with degree 1 vertices

A Bad Graph

A Good Graph

Orient the edges. Label them as follows. Here the inverse edge has opposite orientation.

\[
e_1, e_2, ..., e_{|E|},
\]

\[
e_{|E|+1} = e_1^{-1}, ..., e_{2|E|} = e_{|E|}^{-1}
\]
Primes in Graphs

(correspond to geodesics in compact manifolds)
are equivalence classes $[C]$ of closed backtrackless
tailless primitive paths $C$

DEFINITIONS

backtrack

- equivalence class: change starting point
- tail (backtrack if you change starting vertex)

a path with a backtrack & a tail

non-primitive: go around path more than once

EXAMPLES of Primes in a Graph

$[C] = [e_1e_2e_3]$  
$[D] = [e_4e_5e_3]$  
$[E] = [e_1e_2e_3e_4e_5e_3]$  
$
u(C) = 3, \nu(D) = 3, \nu(E) = 6$

$E = CD$
another prime $[C^oD], \ n = 2, 3, 4, \ldots$
ininitely many primes
Ihara Zeta Function

\[ \zeta_V(u,X) = \prod_{[C] \text{ primes in } X} (1-u^{\nu(c)})^{-1} \]

Ihara's Theorem (Bass, Hashimoto, etc.)

- \( A \) = adjacency matrix of \( X \)
- \( Q \) = diagonal matrix, jth diagonal entry = degree jth vertex -1;
- \( r \) = rank fundamental group = \(|E|-|V|+1\)

\[ \zeta(u,X)^{-1} = (1-u^2)^{-r-1} \det(I-Au+Qu^2) \]

2 Examples

- \( K_4 \) and \( X=K_4\)-edge

\[ \zeta(u,K_4)^{-1} = (1-u^2)^2(1-u)(1-2u)(1+u+2u^2)^3 \]

\[ \zeta(u,X)^{-1} = (1-u^2)(1-u)(1+u^2)(1+u+2u^2)(1-u^2-2u^3) \]
Ihara defined the zeta as a product over p-adic group elements.
Serre saw the graph theory interpretation.
Hashimoto and Bass extended the theory.

- Later we may outline Bass's proof of Ihara's theorem. It involves defining an edge zeta function with more variables.
- Another proof of the Ihara theorem for regular graphs uses the Selberg trace formula on the universal covering tree. For the trivial representation, see A.T., Fourier Analysis on Finite Groups & Applics; for general case, see and Venkov & Nikitin, St. Petersburg Math. J., 5 (1994)

Part of the universal covering tree $T_4$ of a 4-regular graph.

A tree has no closed paths and is connected.

$T_4$ is infinite and so I cannot draw it.

It can be identified with the 3-adic quotient $SL(2, \mathbb{Q}_3)/SL(2, \mathbb{Z}_3)$

A finite 4-regular graph is a quotient of this tree $T_4$ modulo $\Gamma$ the fundamental group of the graph $X$
For a $q+1$-regular graph, meaning that each vertex has $q+1$ edges coming out.

$u=q^{-s}$ makes Ihara zeta more like Riemann zeta.

$f(s) = \zeta(q^{-s})$ has a functional equation relating $f(s)$ and $f(1-s)$.

Riemann Hypothesis (RH) says $\zeta(q^{-s})$ has no poles with $0 < \text{Re } s < 1$ unless $\text{Re } s = \frac{1}{2}$.

RH means graph is Ramanujan, i.e., non-trivial spectrum of adjacency matrix is contained in the spectrum for the universal covering tree which is the interval $(-2\sqrt{q}, 2\sqrt{q})$.

[see Lubotzky, Phillips & Sarnak, Combinatorica, 8 (1988)].

Ramanujan graph is a good expander (good gossip network).

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**What is an expander graph $X$?**

4 Ideas

1) spectral property of some matrix associated to our finite graph $X$
   - Choose one of 3:
     - Adjacency matrix $A$,
     - Laplacian $D - A$, or $I - D^{-1/2}AD^{-1/2}$, $D$=diagonal matrix of degrees
     - edge matrix $W_i$ for $X$ (to be defined)
   - Lubotzky: Spectrum for $X$ SHOULD BE INSIDE spectrum of analogous operator on universal covering tree for $X$.

2) $X$ behaves like a random graph.

3) Information is passed quickly in the gossip network based on $X$.

4) Random walker on $X$ gets lost FAST.
Possible Locations of Poles $u$ of $\zeta(u)$ for $q+1$ Regular Graph

1/q always the closest pole to 0 in absolute value.

$\zeta(\pm \frac{1}{\sqrt{q}})$ from the RH poles.

Real poles ($\neq \pm 1/\sqrt{q}, \pm 1$) correspond to non-RH poles.

Alon conjecture for regular graphs says RH $\equiv$ true for "most" regular graphs. See Joel Friedman's web site for proof (www.math.ubc.ca/~jf) or his Memoirs of the A.M.S., Vol. 195.

See Steven J. Miller's web site at Williams College or Experimental Math. (Volume 17, Issue 2 (2008), 231-244) for experiments leading to conjecture that the percent of regular graphs satisfying RH approaches 27% as number of vertices $\to \infty$, via Tracy-Widom distribution.

Derek Newland's Experiments

Graph analog of Odlyzko experiments for Riemann zeta

Mathematica experiment with random 53-regular graph - 2000 vertices

Spectrum adjacency matrix $\zeta(52-s)$ as a function of $s$

Top row = distributions for eigenvalues of $A$ on left and imaginary parts of the zeta poles on right $s=\frac{1}{2}+it$.

Bottom row = their respective normalized level spacings.

Red line on bottom: Wigner surmise GOE, $y = (\pi x/2) \exp(-\pi x^2/4)$. 
What is the meaning of the RH for irregular graphs?

For irregular graph, natural change of variables is $u=R^2$, where
$R =$ radius of convergence of Dirichlet series for Ihara zeta.
Note: $R$ is closest pole of zeta to 0. No functional equation.
Then the critical strip is $0 \leq \text{Re}u \leq 1$ and translating back to $u$-variable. In the $q+1$-regular case, $R=1/q$.

Graph theory RH:

$\zeta(u)$ is pole free in $R < |u| < \sqrt{R}$

To investigate this, we need to define the edge matrix $W_1$. See Lecture 2.