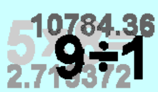
a stroll through the zeta garden

Lecture 1: Riemann, Dedekind, Selberg, and Ihara Zetas



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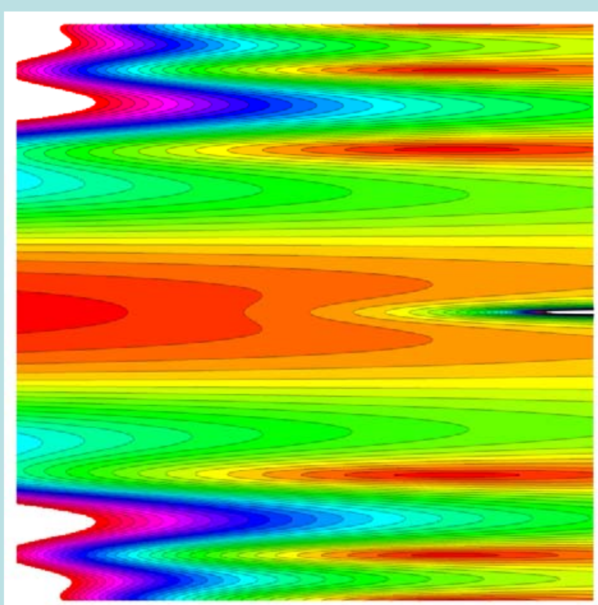


more details can be found in

my webpage:
www.math.ucsd.edu/~aterras/newbook.pdf

First the Riemann Zeta

The **Riemann zeta function** for Re(s) > 1

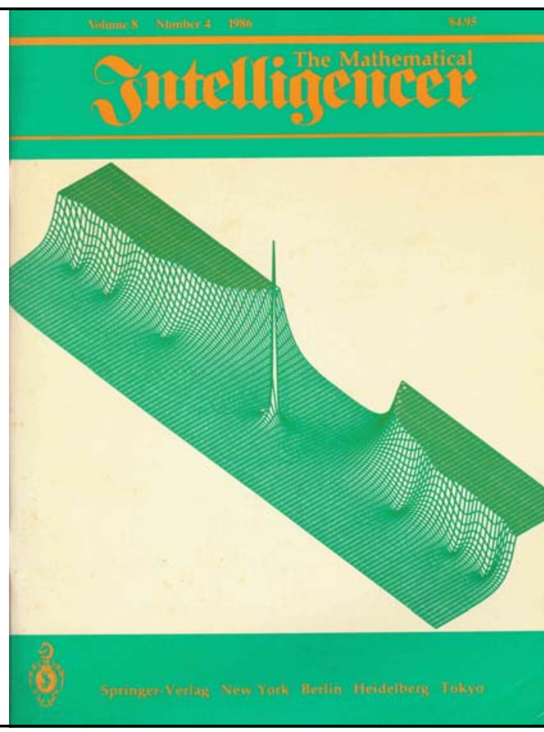
$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p=\text{prime}} \left(1 - p^{-s}\right)^{-1}.$$

- **Riemann (1859) extended to all complex s with pole at s=1**
- **Functional equation relates value at s and 1-s**

$$\Lambda(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s) = \Lambda(1-s)$$

- **Riemann hypothesis (non real zeros $\zeta(s)=0$ are on the line Re(s)=1/2). This now checked for 10^{13} billion zeros. (work of X. Gourdon and P. Demichel). See Ed Pegg Jr.'s website.**

Graph of $z=|\zeta(x+iy)|$ showing the pole at x+iy=1 and the first 6 zeros which are on the line x=1/2, of course. The picture was made by D. Asimov and S. Wagon to accompany their article on the evidence for the Riemann hypothesis as of 1986.



⌘ duality between primes & complex zeros of zeta using Hadamard product over zeros

⌘ **prime number theorem** proved by Hadamard and de la Vallée Poussin (1896-1900). Their proof requires complex analysis

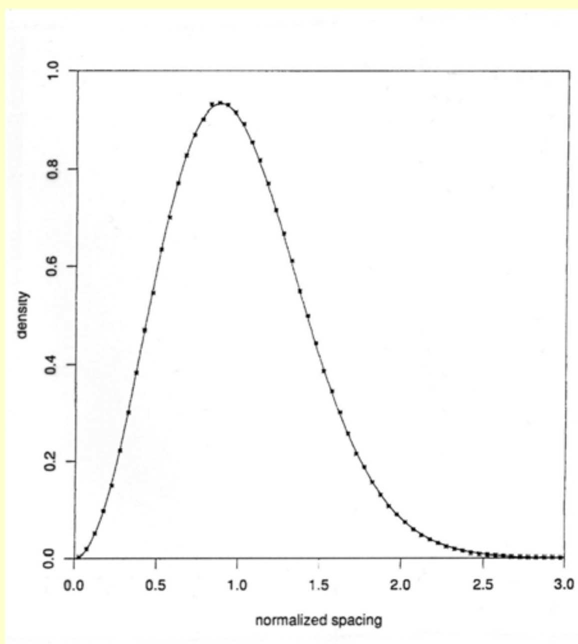
$$\#\{p = \text{prime} \mid p \le x\} \sim \frac{x}{\log x}, \text{ as } x \to \infty$$

⌘ statistics of Riemann zero spacings studied by Odlyzko (GUE)

www.dtc.umn.edu/~odlyzko/doc/zeta.htm

⌘ B. Conrey, The Riemann Hypothesis, Notices, A.M.S., March, 2003

Odlyzko's Comparison of Spacings of 7.8×10^7 Zeros of Zeta at heights $\approx 10^{20}$ & Eigenvalues of Random Hermitian Matrix (GUE).



Many Kinds of Zeta

Dedekind zeta of an algebraic number field F such as $\mathbb{Q}(\sqrt{2})$, where primes become prime ideals p and infinite product of terms $(1-Np^{-s})^{-1}$, where Np = norm of p = #(O/p), O=ring of integers in F

Functional Equations: $\zeta_K(s)$ related to $\zeta_K(1-s)$
Hecke

Values at 0: $\zeta(0) = -1/2$, $\zeta_K(0) = -hR/w$

h = **class number** (measures how far O_K is from having unique factorization) (=1 for $K=\mathbb{Q}(\sqrt{2})$)

R = **regulator** (determinant of logs of units) $= \log(1+\sqrt{2})$ when $K=\mathbb{Q}(\sqrt{2})$

w = **number of roots of unity** in K is 2, when $K=\mathbb{Q}(\sqrt{2})$

Statistics of Prime Ideals and Zeros

- from information on zeros of $\zeta_K(s)$ obtain
  prime ideal theorem

$$\#\{p \text{ prime ideal in } O_K \mid Np \le x\} \sim \frac{x}{\log x}, \text{ as } x \to \infty$$

- there are an infinite number of primes such that $\left(\frac{2}{p}\right)=1$.
- Dirichlet theorem: there are an infinite number of primes p in the progression a, a+d, a+2d, a+3d, ..., when g.c.d.(a,d)=1.
- **Riemannn hypothesis still open:**
  GRH or ERH: $\zeta_K(s)=0$ implies Re(s)=1/2, assuming s is not real.

Selberg zeta associated to a compact Riemannian manifold $M=\Gamma\backslash H$, H = upper half plane with

$ds^2=(dx^2+dy^2)y^{-2}$

Γ =discrete subgroup of group of real fractional linear transformations

primes = primitive closed geodesics *C* in M of length $\nu(C)$, (primitive means only go around once)

$$Z(s)=\prod_{[C]}\prod_{j\geq 0}\left(1-e^{-(s+j)\nu(C)}\right)$$

Duality between spectrum Δ on M &
lengths closed geodesics in M
Z(s+1)/Z(s) is more like Riemann zeta

Realize M as quotient of upper half plane

$H=\{x+iy \mid x,y\in\mathbb{R},\ y>0\}$.

Non-Euclidean distance: $ds^2=y^{-2}(dx^2+dy^2)$

ds is invariant under

$z \to (az+b)/(cz+d)$,

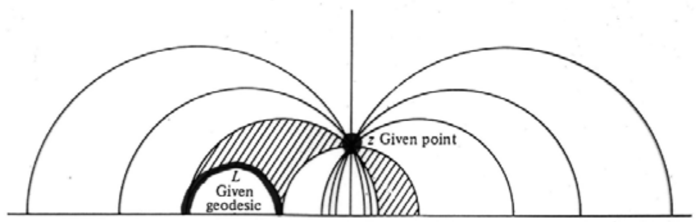
for a,b,c,d real and ad-bc =1. PSL(2,$\mathbb{R}$).

Corresponding Laplacian $\Delta = y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$.

also commutes with action of PSL(2,$\mathbb{R}$).

The curves (geodesics) minimizing arc length are circles and lines in H orthogonal to real axis. Non-Euclidean geometry.

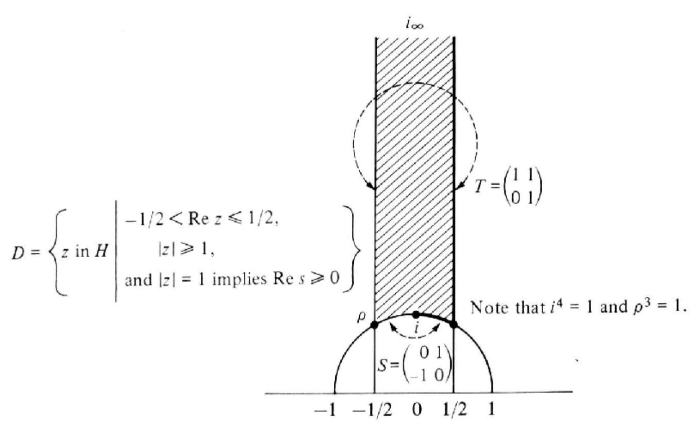
Picture of the Failure of Euclid's 5th Postulate



View compact or finite volume manifold as $\Gamma\backslash H$, where Γ is a discrete subgroup of $PSL(2,\mathbb{R})$. For example,

$\Gamma = PSL(2,\mathbb{Z})$, the modular group.

Fundamental Domain is a non-Euclidean triangle.



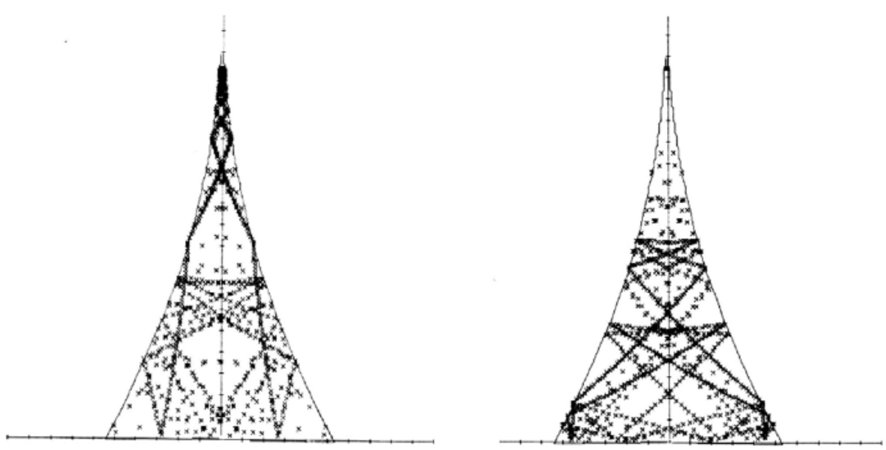
A closed geodesic in $\Gamma\backslash H$ comes from one in H. One can show that the endpoints in $\mathbb{R}$ of such a geodesic are fixed by hyperbolic elements of Γ;

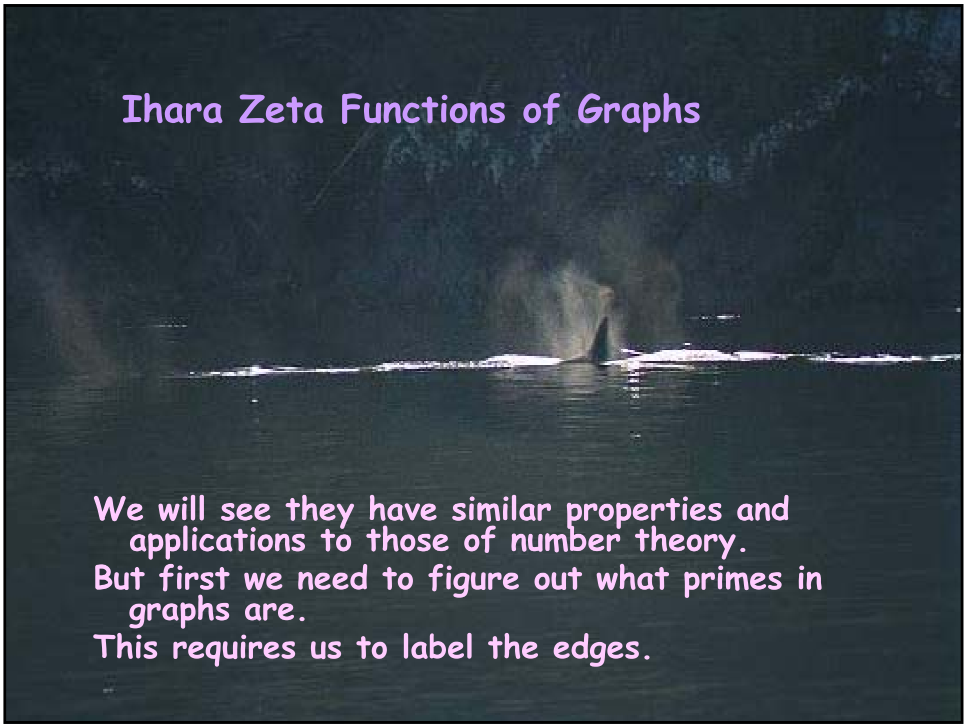
i.e., those $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with trace = a+d>2.

Primitive closed geodesics are traversed only once. They correspond to hyperbolics that generate their centralizer in Γ.
See my book *Harmonic Analysis on Symmetric Spaces*, Vol. I, for more information.

Next a picture of images of points on 2 geodesics circles after mapping them into a fundamental domain of PSL(2,$\mathbb{Z}$)

Images of points on 2 geodesics circles after mapping them into a fundamental domain of PSL(2,$\mathbb{Z}$)

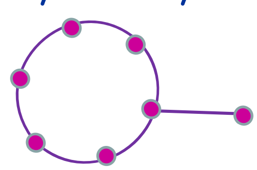




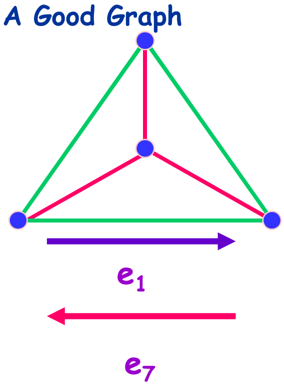
Labeling Edges of Graphs

X = finite connected (not-necessarily regular graph).
Usually we assume: graph is not a cycle or a cycle with degree 1 vertices

A Bad Graph



A Good Graph



Orient the edges. Label them as follows. Here the inverse edge has opposite orientation.

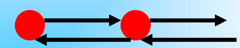
$$e_1, e_2, \ldots, e_{|E|},$$
$$e_{|E|+1} = e_1^{-1}, \ldots, e_{2|E|} = e_{|E|}^{-1}$$

Primes in Graphs

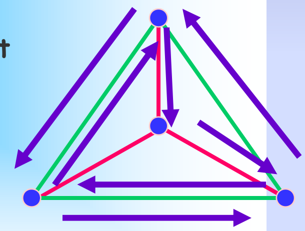
(correspond to geodesics in compact manifolds)

are equivalence classes [*C*] of closed backtrackless tailless primitive paths *C*

DEFINITIONS backtrack



equivalence class: change starting point

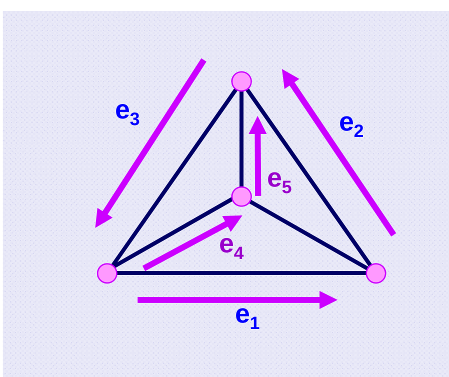


tail (backtrack if you change starting vertex)

a path with a backtrack & a tail

non-primitive: go around path more than once

EXAMPLES of Primes in a Graph



$[C] = [e_1e_2e_3]$

$[D] = [e_4e_5e_3]$

$[E] = [e_1e_2e_3e_4e_5e_3]$

$\nu(C)=3,\ \nu(D)=3,\ \nu(E)=6$

E=CD
another prime $[C^nD]$, n=2,3,4, …
infinitely many primes

Ihara Zeta Function

$$\zeta_V(u,X) = \prod_{[C]\text{ primes in } X} \left(1 - u^{\nu(c)}\right)^{-1}$$

Ihara's Theorem (Bass, Hashimoto, etc.)

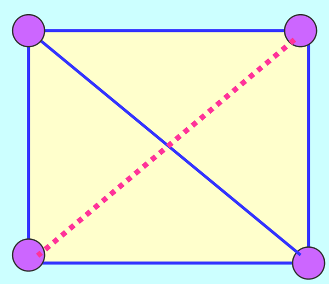
A = adjacency matrix of X

Q = diagonal matrix jth diagonal entry
= degree jth vertex -1;

r = rank fundamental group = |E|-|V|+1

$$\zeta(u,X)^{-1} = (1-u^2)^{r-1}\det(I - Au + Qu^2)$$

2 Examples
K_4 and
$X=K_4$-edge



$$\zeta(u,K_4)^{-1} = (1-u^2)^2(1-u)(1-2u)(1+u+2u^2)^3$$

$$\zeta(u,X)^{-1} = (1-u^2)(1-u)(1+u^2)(1+u+2u^2)(1-u^2-2u^3)$$

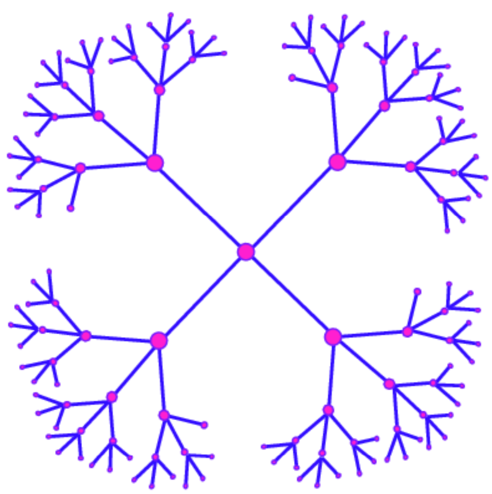
Remarks

Ihara defined the zeta as a product over p-adic group elements.

Serre saw the graph theory interpretation.

Hashimoto and Bass extended the theory.

- Later we may outline Bass's proof of Ihara's theorem. It involves defining an edge zeta function with more variables
- Another proof of the Ihara theorem for regular graphs uses the Selberg trace formula on the universal covering tree. For the trivial representation, see A.T., *Fourier Analysis on Finite Groups & Applics;* for general case, see and Venkov & Nikitin, *St. Petersberg Math. J.*, 5 (1994)



Part of the universal covering tree T_4 of a 4-regular graph.

A tree has no closed paths and is connected.

T_4 is infinite and so I cannot draw it.

It can be identified with the 3-adic quotient $SL(2,\mathbb{Q}_3)/SL(2,\mathbb{Z}_3)$

A finite 4-regular graph is a quotient of this tree T_4 modulo Γ=the fundamental group of the graph X

For $q+1$ – regular graph, meaning that each vertex has $q+1$ edges coming out

$u=q^{-s}$ makes Ihara zeta more like Riemann zeta.

$f(s)=\zeta(q^{-s})$ has a functional equation relating $f(s)$ and $f(1-s)$.

Riemann Hypothesis (RH)

says $\zeta(q^{-s})$ has no poles with $0<\mathrm{Re}\,s<1$ unless $\mathrm{Re}\,s = \frac{1}{2}$.

RH means graph is Ramanujan i.e., non-trivial spectrum of adjacency matrix is contained in the spectrum for the universal covering tree which is the interval $(-2\sqrt{q}, 2\sqrt{q})$

[see Lubotzky, Phillips & Sarnak, *Combinatorica*, 8 (1988)].

Ramanujan graph is a good expander (good gossip network)

What is an expander graph X?

4 Ideas

1) spectral property of some matrix associated to our finite graph X
   Choose on of 3:
- Adjacency matrix A,
- Laplacian D-A, or $I-D^{-1/2}AD^{-1/2}$, D=diagonal matrix of degrees
- edge matrix W_1 for X (to be defined)

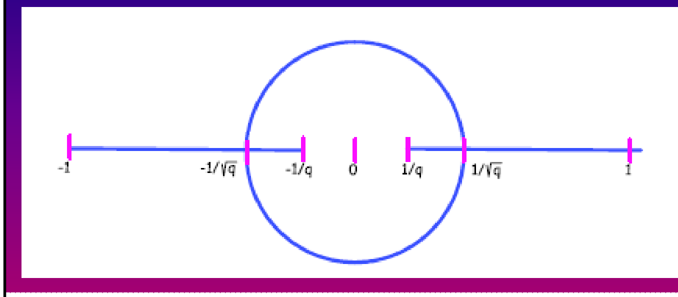
   Lubotzky: Spectrum for X SHOULD BE INSIDE spectrum of analogous operator on universal covering tree for X.

2) X behaves like a random graph.

3) Information is passed quickly in the gossip network based on X.

4) Random walker on X gets lost FAST.

Possible Locations of Poles u of ζ(u) for q+1 Regular Graph



1/q always the closest pole to 0 in absolute value.

Circle of radius $1/\sqrt{q}$ from the RH poles.

Real poles ($\neq \pm q^{-1/2}$, ± 1) correspond to non-RH poles.

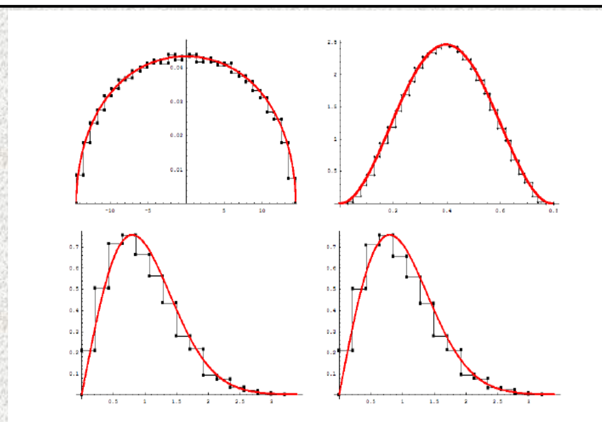
Alon conjecture for regular graphs says RH ≅ true for "most" regular graphs. See Joel Friedman's web site for proof (www.math.ubc.ca/~jf) or his Memoirs of the A.M.S., Vol. 195.

See Steven J. Miller's web site at Williams College or Experimental Math. (Volume 17, Issue 2 (2008), 231-244) for experiments leading to conjecture that the percent of regular graphs satisfying RH approaches 27% as # vertices → ∞, via Tracy-Widom distribution.

Derek Newland's Experiments

Graph analog of Odlyzko experiments for Riemann zeta

Mathematica experiment with random 53-regular graph - 2000 vertices



Spectrum adjacency matrix | $\zeta(52^{-s})$ as a function of s

Top row = distributions for eigenvalues of A on left and imaginary parts of the zeta poles on right $s=\frac{1}{2}+it$.

Bottom row = their respective normalized level spacings.

Red line on bottom: Wigner surmise GOE, $y = (\pi x/2)\exp(-\pi x^2/4)$.

What is the meaning of the RH for irregular graphs?

For irregular graph, natural change of variables is $u=R^s$, where

R = radius of convergence of Dirichlet series for Ihara zeta.

Note: R is closest pole of zeta to 0. No functional equation.

Then the critical strip is $0 \le \mathrm{Re}\,s \le 1$ and translating back to u-variable. In the q+1-regular case, $R=1/q$.

Graph theory RH:

$\zeta(u)$ is pole free in $R < |u| < \sqrt{R}$

To investigate this, we need to define the edge matrix W_1. See Lecture 2.