### Example 1. Quadratic Extension

<table>
<thead>
<tr>
<th>field</th>
<th>ring</th>
<th>prime ideal</th>
<th>finite field</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K = \mathbb{F}(\sqrt{m})$</td>
<td>$O_K = \mathbb{Z}[[\sqrt{m}]]$</td>
<td>$\mathfrak{p} \supset O_K$</td>
<td>$O_K / \mathfrak{p}$</td>
</tr>
<tr>
<td>$\mathbb{F} = \mathbb{Q}$</td>
<td>$O_F = \mathbb{Z}$</td>
<td>$p\mathbb{Z}$</td>
<td>$\mathbb{Z} / p\mathbb{Z}$</td>
</tr>
</tbody>
</table>

$g = \#$ of such $\mathfrak{p}$, $f =$ degree of $O_K / \mathfrak{p}$ over $O_F / pO_F$, $p^e \supset pO_K \not\subset p^{e+1}$

$efg = 2$

Assume, $m$ is a square-free integer congruent to 2 or 3 (mod 4).

Decomposition of Primes in Quadratic Extensions

3 CASES

1) **p inert:** $f=2$. 
$p_{OK} = \text{prime ideal in } K, \quad m \not\equiv x^2 \pmod{p}$

2) **p splits:** $g=2$. 
$p_{OK} = p \cdot p', \quad p \neq p', \quad m \equiv x^2 \pmod{p}$

3) **p ramifies:** $e=2$. 
$p_{OK} = p^2, \quad p \text{ divides } 4m$

$Gal(K/F) = \{1,-1\}$

Frobenius automorphism

$\sigma = \tfrac{4m}{p}$

<table>
<thead>
<tr>
<th>$\tfrac{4m}{p}$</th>
<th>Case 1</th>
<th>Case 2</th>
<th>Case 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 \quad \text{ in case 1}</td>
<td>1 \quad \text{ in case 2}</td>
<td>0 \quad \text{ in case 3}</td>
<td></td>
</tr>
</tbody>
</table>

$p \text{ does not divide } 4m$ implies $p$ has 50% chance of being in Case 1 (and 50% chance of being in case 2)

Assume, $m$ is a square-free integer $\equiv 2$ or $3 \pmod{4}$.

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**Artin L-Functions**

$K \supset F$ number fields with $K/F$ Galois

$O_K \supset O_F$ rings of integers

$\mathfrak{P} \supset \mathfrak{p}$ prime ideals ($\mathfrak{p}$ unramified, i.e., $\mathfrak{p} \not\subset \mathfrak{P}^2$)

Frobenius Automorphism when $\mathfrak{p}$ is unramified.

$\left(\frac{K/F}{\mathfrak{p}}\right) = \sigma_{\mathfrak{p}} \in Gal(K/F), \quad \sigma_{\mathfrak{p}}(x) \equiv x^{\mathfrak{p}^f/\mathfrak{p}^e} \pmod{\mathfrak{P}^r}$, for $x \in O_K$

$\sigma_{\mathfrak{p}}$ induces generator of finite Galois group, $Gal((O_K/\mathfrak{P})/(O_F/\mathfrak{p}))$ determined by $\mathfrak{p}$ up to conjugation if $\mathfrak{P}/\mathfrak{p}$ unramified $f (\mathfrak{P}/\mathfrak{p}) = \text{order of } \sigma_{\mathfrak{p}} = [O_K/\mathfrak{P} : O_F/\mathfrak{p}]$

$g (\mathfrak{P}/\mathfrak{p}) = \text{number of } \text{primes of } K \text{ dividing } \mathfrak{p}$

Artin L-Function for $s \in \mathbb{C}$, $\pi$ a representation of $Gal(K/F)$.

Give only the formula for unramified primes $\mathfrak{p}$ of $F$.

Pick $\mathfrak{P}$ a prime in $O_K$ dividing $\mathfrak{p}$.

$L(s, \pi)'' = \prod_{\mathfrak{p}} \det \left(1 - \pi \left(\frac{K/F}{\mathfrak{P}}\right)^N \mathfrak{p}^{-s}\right)^{-1}$
For a set $S$ of primes of $F$, define the analytic density of $S$.

$$\delta(S) = \lim_{s \to 1^+} \left( \frac{\sum_{p \in S} p^{-s}}{-\log(s-1)} \right)$$

**Theorem.** Set $C(p)$ = the conjugacy class of the Frobenius automorphism of prime ideals $P$ of $K$ above $p$. Then, for every conjugacy class $C$ in $G = \text{Gal}(K/F)$,

$$\delta\{ p \mid C(p) = C \} = \frac{|C|}{|G|}.$$

The proof requires the facts that $L(s,\pi)$ continues to $s=1$ with no pole or zero if $\pi \neq 1$, while $L(s,1) = \zeta_F(s)$ has a simple pole at $s=1$.

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**Graph Galois Theory**

Graph $Y$ an unramified covering of Graph $X$ means (assuming no loops or multiple edges)

$$\exists \, \pi: Y \to X \text{ is an onto graph map such that for every } x \in X \text{ & for every } y \in \pi^{-1}(x),$$

$$\pi \text{ maps the points } z \in Y \text{ adjacent to } y 1-1, \text{ onto the points } w \in X \text{ adjacent to } x.$$

**Normal d-sheeted Covering** means:

$$\exists d \text{ graph isomorphisms } g_1, \ldots, g_d \text{ mapping } Y \to Y$$

such that

$$\pi \circ g_i (y) = \pi (y) \quad \forall \, y \in Y$$

The Galois group

$$G(Y/X) = \{ g_1, \ldots, g_d \}.$$

This is an analog of coverings of manifolds, Riemann surfaces, etc.
First pick a spanning tree $T_X$ in $X$ (no cycles, connected, includes all vertices of $X$).

Second make $n=|G|$ copies of the tree $T_X$ in $X$. These are the sheets of $Y$. Label the sheets with $g \in G$. The vertices of $Y$ are $(x, g)$ for $x$ vertex of $X, g \in G$.

$G$ action on $Y$:
- on sheets: $g(\text{sheet } h) = \text{sheet}(gh)$
- on vertices: $g(x, h) = (x, gh)$
- on paths: $g(\text{path from } (v, h) \text{ to } (w, j)) = \text{path from } (v, gh) \text{ to } (w, gj)$

Given $G$, get examples $Y$ by giving permutation representation of generators of $G$ to lift edges of $X$ left out of $T_X$.

Example 1. Quadratic Cover

Cube covers Tetrahedron

Spanning Tree in $X$ is red. Corresponding sheets of $Y$ are also red
Example of Splitting of Primes in Quadratic Cover

Example of Splitting of Primes in Quadratic Cover

Picture of Splitting of Prime which is inert; i.e., $f=2$, $g=1$, $e=1$
1 prime cycle $D$ above, & $D$ is lift of $C^2$.

Picture of Splitting of Prime which splits completely; i.e., $f=1$, $g=2$, $e=1$
2 primes cycles above
Frobenius Automorphism

\( Frob(D) = \left( \frac{Y/X}{D} \right) = ji^{-1} \in G = \text{Gal}(Y/X) \)

where \( ji^{-1} \) maps sheet \( i \) to sheet \( j \)

\( \tilde{C} = \) the unique lift of \( C \) in \( Y \) starting at \((v,i)\) ending at \((v,j)\)

\( \tilde{C} \) is not necessarily closed

\( \text{length} (\tilde{C}) = \text{length} (C) \)

(\( D \) a prime above \( C \) is closed and is obtained by \( f \) liftings like \( \tilde{C} \))

Exercise: Compute \( Frob(D) \) on preceding pages, \( G = \{e, g\} \).

Galois Group=\( \{e, g\} \): Label cube vertices

\( (x,e) \rightarrow x' \) and \( (x,g) \rightarrow x'' \), \( x \) in \( K_4 \)

Frobenius of prime in \( X \) = non-trivial element of Galois group since

if we lift path on \( X \) once, we get to the other sheet of the cover
**Properties of Frobenius**

1) Replace \((v,i)\) with \((v,hi)\). Then \(\text{Frob}(D) = ji^{-1}\) is replaced with \(hji^{-1}h^{-1}\). Or replace \(D\) with different prime above \(C\) and see that Conjugacy class of \(\text{Frob}(D) \in \text{Gal}(Y/X)\) unchanged.

2) Varying start vertex \(v\) of \(C\) in \(X\) does not change \(\text{Frob}(D)\).

3) \(\text{Frob}(D)^i = \text{Frob}(D^i)\).

**Artin L-Function**

\[ L(u, \rho, Y/X) = \prod_{[C]} \det \left( 1 - \rho \left( \frac{Y/X}{D} \right) u^{v(C)} \right)^{-1} \]

\([C]=\text{equivalence class of primes of } X\]

\(v(C)=\text{length } C, \ D \text{ a prime in } Y \text{ over } C\)

**Question:** How does the Frobenius depend on the labeling, choice of spanning tree, etc.?

**Answer:** You can identify the Galois group \(G(Y/X)\) with a quotient \(\Gamma/H\), \(\Gamma\) the fundamental group of \(X\), a group which can be viewed as generated by the edges left out of a spanning tree.
Properties of Artin L-Functions

1) $L(u, 1, Y/X) = \zeta(u, X)$ = Ihara zeta function of $X$ (our analog of the Dedekind zeta function, also Selberg zeta)

Proof by Defn.

2) $\zeta(u, Y) = \prod_{\rho \in \hat{G}} L(u, \rho, Y / X)^{d_{\rho}}$

product over all irreducible reps of $G$, $d_{\rho}=$degree $\rho$.

Proof uses induced representations and decomposition

$\text{Ind}_{\{e\}}^G 1 = \sum_{\pi \in \hat{G}} d_{\pi} \pi$

See A. T., Fourier Analysis on Finite Groups and Applications.

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Det(I-uW_1) formula for Artin L-Functions

Set $B=W_1$ and call the Frobenius automorphism of an edge $\text{Frob}(e)$. Define the blocks of the matrix $2|E|d_{\rho} \times 2|E|d_{\rho}$ matrix $B_{\rho}$ as follows, for each pair of oriented edges $e,f$ in $X$:

$\left( B_{\rho} \right)_{ef} = (b_{ef} \rho(\text{Frob}(e)))$

$L(u, \rho, Y / X)^{-1} = \det(I - uB_{\rho})$
For the cube over $K_4$ we have 2 degree 1 representations of the Galois group. The only interesting matrix is that for the non-trivial representation: 12x12 matrix. It is too big to put on a Power Point talk or blackboard.

Ihara Theorem for L-Functions

$$L(u, \rho, Y / X)^{-1} = (1 - u^2)^{(r-1)d} \det(I' - A'_\rho u + Q'u^2)$$

- $r=$ rank fundamental group of $X = |E| - |V| + 1$
- $\rho=$ representation of $G = \text{Gal}(Y/X)$, $d = d_\rho =$ degree $\rho$

Definitions. $n \times n$ matrices $A'$, $Q'$, $I'$, $n = |X|$, $n \times n$ matrix $A(g)$, $g \in \text{Gal}(Y/X)$, entry for $a,b$ vertices in $X$ $(A(g))_{a,b} = \#\{\text{edges in } Y \text{ from } (a,e) \text{ to } (b,g)\}$, $e=$identity $\in G$.

$$A'_\rho = \sum_{g \in G} A(g) \otimes \rho(g)$$

- $Q =$ diagonal matrix, $j$th diagonal entry $q_j = (\text{degree of } j\text{th vertex in } X)^{-1}$
- $Q' = Q \otimes I_d$, $I' = I_{nd} =$ identity matrix.
\[ B \otimes C = \begin{pmatrix} b_{11}C & \cdots & b_{1p}C \\ \vdots & \ddots & \vdots \\ b_{p1}C & \cdots & b_{pp}C \end{pmatrix}. \]

**B is pxp and C is qxq**

\[ B \otimes C \text{ is } pqxpq \]

**Y=cube, X=tetrahedron: \( G = \{e, g\} \)**

representations of \( G \) are 1 and \( \rho \):
\[
\rho(e) = 1, \quad \rho(g) = -1
\]

\[ A(e)_{u,v} = \#\{ \text{length 1 paths } u' \text{ to } v' \text{ in } Y\} \]
\[ A(g)_{u,v} = \#\{ \text{length 1 paths } u' \text{ to } v'' \text{ in } Y\} \]

\[ A'_{1} = A = \text{adjacency matrix of } X = A(e) + A(g) \]

\[ A(e) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad A(g) = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \]

\[ A'_{\rho} = A(e) - A(g) = \begin{pmatrix} 0 & 1 & -1 & -1 \\ 1 & 0 & 1 & 1 \\ -1 & 1 & 0 & -1 \\ -1 & 1 & -1 & 0 \end{pmatrix} \]

**EXAMPLE**

\[ (u,e)=u', \quad (u,g) \]

\[ A_{e} \]
Zeta and L-Functions of Cube & Tetrahedron

\(X = K_4\) and \(Y = \text{cube}\)

\[\zeta(u,X)^{-1} = (1-u^2)^2 (1-u)(1-2u) (1+u+2u^2)^3\]

\[L(u,\rho,Y/X)^{-1} = (1-u^2) (1+u) (1+2u) (1-u+2u^2)^3\]

\[\zeta(u,Y)^{-1} = L(u,\rho,Y/X)^{-1} \zeta(u,X)^{-1}\]

Get L function of \(\zeta(u,X)\) by replacing \(u\) by \(-u\) for this example.

Example

Galois Cover of Non-Normal Cubic

\(G = S_3,\ H = \{(1),(23)\}\) fixes \(Y_3\).

\(a^{(1)} = (a,(1)),\ a^{(2)} = (a,(13)),\ a^{(3)} = (a,(132))\)

\(a^{(4)} = (a,(23)), a^{(5)} = (a,(123)), a^{(6)} = (a,(12)).\)

Here we use standard cycle notation for elements of the symmetric group.
Prime Splitting Completely

path in X (list vertices) 14312412431

f=1, g=3  3 lifts to Y₃

Frobenius trivial \Rightarrow density 1/6

This is an analog of the prime 31 for \( \mathbb{Q}(2^{1/3}) \) in Stark's article in From Number Theory to Physics, M. Waldschmidt et al (Eds.), Springer-Verlag, Berlin, 1992, pages 313-393.

Ihara Zeta Functions

\[ \zeta(u,X)^{-1} = (1-u^2)(1-u)(1+u^2)(1+u+2u^2)(1-u^2-2u^3) \]

\[ \zeta(u,Y_3)^{-1} = \zeta(u,X)^{-1} * (1-u^2)^2(1-u-u^3+2u^4) * (1+u+2u^2+u^3+2u^4)(1-u+2u^2-u^3+2u^4)(1+u^3+2u^4) \]

\[ \zeta(u,Y_6)^{-1} = \zeta(u,Y_3)^{-1} (1-u^2)^3 (1+u)(1+u^2)(1-u+2u^2)(1-u+2u^3) * (1-u^3+2u^4) (1-u+2u^2-u^3+2u^4) * (1+u+u^3+2u^4)(1+u+2u^2+u^3+2u^4) \]

It follows that, as in number theory

\[ \zeta(u,X)^2 \zeta(u,Y_6) = \zeta(u,Y_2) \zeta(u,Y_3)^2 \]

\( Y_2 \) is an intermediate quadratic extension between \( Y_6 \) and \( X \).

See Stark & Terras, Adv. in Math., 154 (2000), Fig. 13, for more info.
You can't hear the shape of a graph.

2 connected regular graphs (without loops & multiple edges) which are isospectral but not isomorphic

See A.T. & Stark in Adv. in Math., Vol. 154 (2000) for the details. The method goes back to algebraic number theorists who found number fields \( K \), which are non-isomorphic but have the same Dedekind zeta.


Robert Perlis and Aubi Mellein have used the same methods to find many examples of isospectral non-isomorphic graphs with multiple edges and components. 2 such are on the right.
There are nice constructions of isospectral non-isomorphic graphs in paper of Lubotzky, Samuels and Vishne.

Homework Problems

What are ramified coverings of graphs? Do the zetas⁻¹ divide?

Is there a graph analog of regulator, Stark Conjectures, class field theory for abelian graph coverings? Or more simply a quadratic reciprocity law, fundamental units? The ideal class group is the Jacobian of a graph and has order = number of spanning trees (paper of Roland Bacher, Pierre de la Harpe and Tatiana Nagnibeda). There is an analog of Brauer-Siegel theory (see H.S. and A.T. , Part III).


Beth Malmskog & Michelle Manes, Almost divisibility in the Ihara zeta functions of certain ramified covers of q+1-regular graphs, preprint.