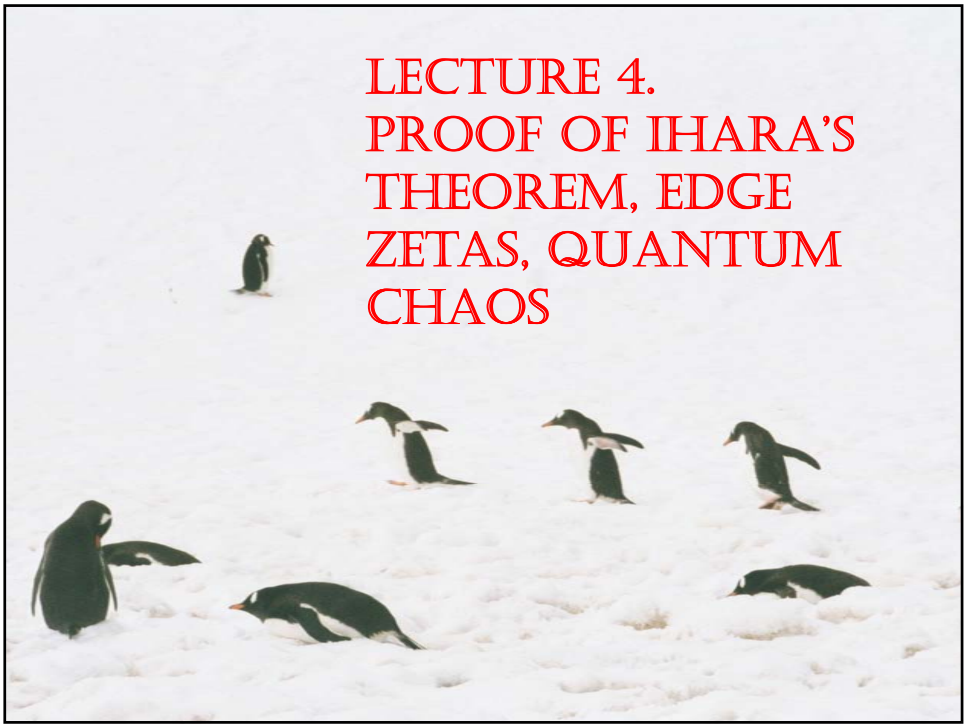


Lecture 4. Proof of Ihara's Theorem, Edge Zetas, Quantum Chaos

Ihara Zeta Function

$$\zeta(u,X) = \prod_{\substack{[C] \\ \text{prime}}} \left(1 - u^{\nu(C)}\right)^{-1}$$

$\nu(C)$ = # edges in *C*
converges for u complex, |u| small

Ihara's Theorem.

$$\zeta(u,X)^{-1} = (1-u^2)^{r-1} \det(I - Au + Qu^2)$$

A=adjacency matrix, Q +I = diagonal matrix of degrees, r=rank fundamental group.

Edge Zetas

Orient the edges of the graph. Multiedge matrix W has ab entry w_{ab} in $\mathbb{C}$, $w(a,b)=w_{ab}$ if the edges a and b look like

$$\bullet \xrightarrow{a} \bullet \xrightarrow{b} \bullet$$

and a is not the inverse of b

Otherwise set $w_{ab}=0$.

For a prime $C = a_1 a_2 \ldots a_s$, define the edge norm

$$N_E(C) = w(a_s,a_1)w(a_1,a_2)w(a_2,a_3)\cdots w(a_{s-1},a_s)$$

Define the edge zeta for small $|w_{ab}|$ as

$$\zeta_E(W,X) = \prod_{[C]} \left(1 - N_E(C)\right)^{-1}$$

Properties of Edge Zeta

⌘ Ihara $\zeta(u,X) = \zeta_E(W,X)|_{\text{non-0 } w(i,j)=u}$

⌘ edge e deletion

$\zeta_E(W,X-e) = \zeta_E(W,X)|_{0=w(i,j)}$, if i or j=e

Determinant Formula For Edge Zeta

$$\zeta_E(W,X) = \det\left(I - W\right)^{-1}$$

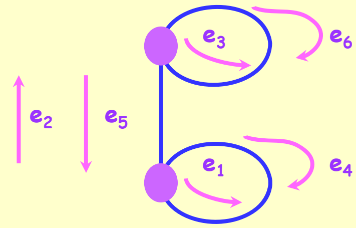
From this Bass gives an ingenious proof of Ihara's theorem.

Reference:

Stark and T., *Adv. in Math.*, Vol. 121 and 154 and 208 (1996 and 2000 and 2007)

Example

D=Dumbbell Graph



$$\zeta_E(W,D)^{-1} = \det\begin{pmatrix} w_{11}-1 & w_{12} & 0 & 0 & 0 & 0 \\ 0 & -1 & w_{23} & 0 & 0 & w_{26} \\ 0 & 0 & w_{33}-1 & 0 & w_{35} & 0 \\ 0 & w_{42} & 0 & w_{44}-1 & 0 & 0 \\ w_{51} & 0 & 0 & w_{54} & -1 & 0 \\ 0 & 0 & 0 & 0 & w_{65} & w_{66}-1 \end{pmatrix}$$

e_2 and e_5 are the vertical edges.

Specialize all variables with 2 and 5 to be 0 and get zeta function of subgraph with vertical edge removed. Fission

Diagonalizes the matrix.

Proof of the Determinant Formula

$$\log \zeta_E^{-1} = \sum_{[P]} \sum_{j\geq 1} \frac{N_E(P)^j}{j}$$

$\nu(P)=m \Rightarrow \#[P]=m.$

$$\log \zeta_E^{-1} = \sum_{m\geq 1} \frac{1}{m} \sum_{\substack{P \\ \nu(P)=m}} \sum_{k\geq 1} \frac{N_E(P)^k}{k}$$

P prime path, not class [P]

non-prime paths C are powers of primes $C=P^k$

$$\log \zeta_E^{-1} = \sum_C \frac{1}{\nu(C)} N_E(C) = \sum_{m\geq 1} \frac{1}{m} Tr(W^m)$$

Here C need not be prime path, still closed, no backtrack, no tails

for last = use the same sort of argument as in Lecture 2.

End of Proof

Using the matrix calculus exercise from Lecture 2

det(exp(B)=exp(Tr(B)) gives

$$\log \zeta_E(W)^{-1} = \sum_{m \geq 1} \frac{1}{m} Tr(W^m) = \log \det\left(I - W\right)^{-1}$$

This proves (log(determinant formula)).

$$\zeta_E(W, X) = \det\left(I - W\right)^{-1}$$

Bass Proof of Ihara formula

$$\zeta_E(W, X) = \det\left(I - W\right)^{-1}$$

$$\zeta_V(u, X)^{-1} = (1-u^2)^{r-1} \det(I - Au + Qu^2)$$

Part 1 of Bass Proof

Define

$$J=\begin{pmatrix}0 & I_{|E|}\\ I_{|E|} & 0\end{pmatrix}$$

Define starting matrix S and terminal matrix T
- both |V|x2|E| matrices of 0s and 1s

$$s_{ve}=\begin{cases}1, \text{ if v is starting vertex of oriented edge e}\\ 0, \text{ otherwise}\end{cases}$$

$$t_{ve}=\begin{cases}1, \text{ if v is the terminal vertex of oriented edge e}\\ 0, \text{ otherwise}\end{cases}$$

Then, recalling our edge numbering system, we see that

$$SJ=T, \qquad \mathrm{TJ=S}$$

since start (end) of e_j is end (start) of $e_{j+|E|}$

$$A=S\ \mathrm{T}^t, \quad \mathrm{Q+I}_{|V|}=SS^t=TT^t$$

Note: matrix A counts number of undirected edges connecting 2 distinct vertices and twice # of loops at each vertex. Q+I = diagonal matrix of degrees of vertices

Part 2 of Bass Proof

W_1 matrix obtained from W by setting all non-zero w_{ij} equal to 1

$W_1 + J = T^t S$, where J compensates for not allowing edge e_j to feed into $e_{j\pm|E|}$

Below all matrices are (|V|+2|E|) x (|V|+2|E|), with |V| x |V| 1st block.

The preceding formulas imply that:

$$\begin{pmatrix}I_{|V|} & 0\\ T^t & I_{2|E|}\end{pmatrix}\begin{pmatrix}I_{|V|}(1-u^2) & Su\\ 0 & I_{2|E|}-W_1u\end{pmatrix}$$

$$=\begin{pmatrix}I_{|V|}-Au+Qu^2 & Su\\ 0 & I_{2|E|}+Ju\end{pmatrix}\begin{pmatrix}I_{|V|} & 0\\ T^t-S^tu & I_{2|E|}\end{pmatrix}$$

Then take determinants of both sides to see

$$(1-u^2)^{|V|}\det(I_{2|E|}-W_1u)=\det(I_{|V|}-Au+Qu^2)\det(I_{2|E|}+Ju)$$

End of Bass Proof

$$(1-u^2)^{|V|}\det(I_{2|E|}-W_1u)=\det(I_{|V|}-Au+Qu^2)\det(I_{2|E|}+Ju)$$

$$I+Ju=\begin{pmatrix} I & Iu \\ Iu & I \end{pmatrix} \text{ implies } \begin{pmatrix} \mathrm{I} & 0 \\ \text{-Iu} & \mathrm{I} \end{pmatrix}(I+Ju)=\begin{pmatrix} I & Iu \\ 0 & I(1-u^2) \end{pmatrix}$$

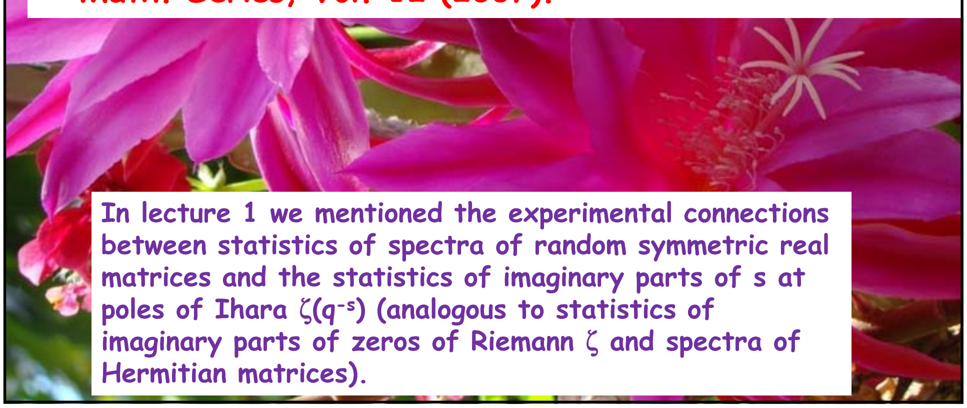
So $\det(I+Ju)=(1-u^2)^{|E|}$

Since r-1=|E|-|V|, for a connected graph, the Ihara formula for the vertex zeta function follows from the edge zeta determinant formula.

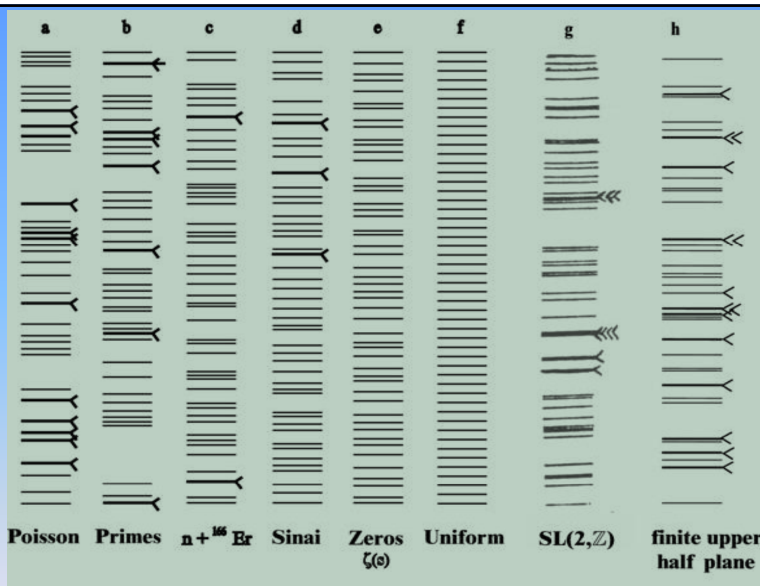
A Taste of Random Matrix Theory / Quantum Chaos

a reference with some background on the interest in random matrices in number theory and quantum physics:

A.Terras, Arithmetical quantum chaos, IAS/Park City Math. Series, Vol. 12 (2007).



In lecture 1 we mentioned the experimental connections between statistics of spectra of random symmetric real matrices and the statistics of imaginary parts of s at poles of Ihara $\zeta(q^{-s})$ (analogous to statistics of imaginary parts of zeros of Riemann ζ and spectra of Hermitian matrices).



from O. Bohigas and M.-J. Giannoni, Chaotic motion and random matrix theories, *Lecture Notes in Physics*, 209, Springer-Verlag, Berlin, 1984: arrows mean lines are too close to distiguish

The dichotomy

RMT spacings (GOE etc) $\frac{1}{2}\pi x\exp(-\pi x^2/4)$	Poisson spacings $\exp(-x)$
quantum spectra of system with chaos in classical counterpart Bohigas Giannoni Schmit Conjecture, 1984	energy levels of quantum system with integrable system for classical counterpart Berry Tabor Gutzwiller Conjecture, 1977
eigenvalues of Laplacian for non-arithmetic manifold	eigenvalues of Laplacian for arithmetic manifold; e.g. $H/SL(2,\mathbb{Z})$
zeros Riemann zeta	

Sarnak invented the term "arithmetical quantum chaos" to describe the 2nd row of our table. See the book of Katz and Sarnak for some proved results about zetas of curves over finite fields.

See Rudnick "What is quantum chaos?" Notices AMS, Jan. 2008 for definitions of some of these things in the context of billiards. None of these conjectures is proved as far as I know.

Now we wish to add a new column to earlier figure
- spacings of the eigenvalues of the W_1 matrix of a graph

Here although W_1 is not symmetric, we mean the nearest neighbor spacing (i.e., histogram of minimum distances between eigenvalues which lie in the complex plane not just the real axis).

Reference on spacings of spectra of non-Hermitian or non-symmetric matrices.
P. LeBoef, Random matrices, random polynomials, and Coulomb systems, ArXiv.
J. Ginibre, J. Math. Phys. 6, 440 (1965).
Mehta, Random Matrices, Chapter 15.

An approximation to the density for spacings of eigenvalues of a complex matrix (analogous to the Wigner surmise for Hermitian matrices) is:

$$4\Gamma\left(\frac{5}{4}\right)^4 s^3 e^{-\Gamma\left(\frac{5}{4}\right)^4 s^4}$$

Statistics of the poles of Ihara zeta or reciprocals of eigenvalues of the Edge Matrix W_1

Define W_1 to be the 0,1 matrix you get from W by setting all non-0 entries of W to be 1.

Theorem. $\zeta(u,X)^{-1}=\det(I-W_1 u)$.

Corollary. The poles of Ihara zeta are the reciprocals of the eigenvalues of W_1.

The pole R of zeta is:

R=1/Perron-Frobenius eigenvalue of W_1.

Properties of W_1

1) $W_1 = \begin{pmatrix} A & B \\ C & A^T \end{pmatrix}$ **B and C symmetric real, A real**

2) **jth row sums of entries are q_j+1=degree vertex which is start of edge j.**

Poles Ihara Zeta are in region $q^{-1} \le R \le |u| \le 1$,
$q+1$=maximum degree of vertices of X.

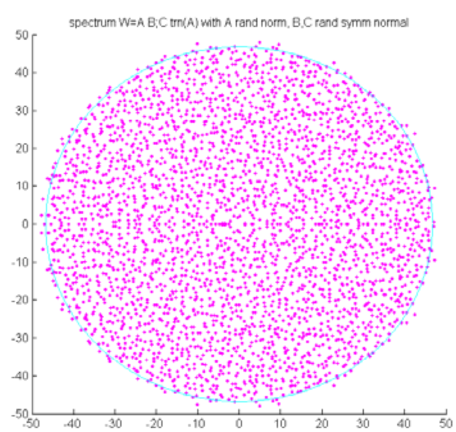
Theorem of Kotani and Sunada

If $p+1$=min vertex degree, and $q+1$=maximum vertex degree, non-real poles u of zeta satisfy

$$\frac{1}{\sqrt{q}} \le |u| \le \frac{1}{\sqrt{p}}$$

Kotani & Sunada, *J. Math. Soc. U. Tokyo*, 7 (2000)
or see my manuscript on my website:
www.math.ucsd.edu\~aterras\newbook.pdf

Spectrum of Random Matrix with Properties of W -matrix



$$W = \begin{pmatrix} A & B \\ C & A^T \end{pmatrix}$$

entries of **W** are non-negative from normal distribution

B and C symmetric diagonal entries are 0

Girko circle law for real matrices with circle of radius ½(1+√2) √n

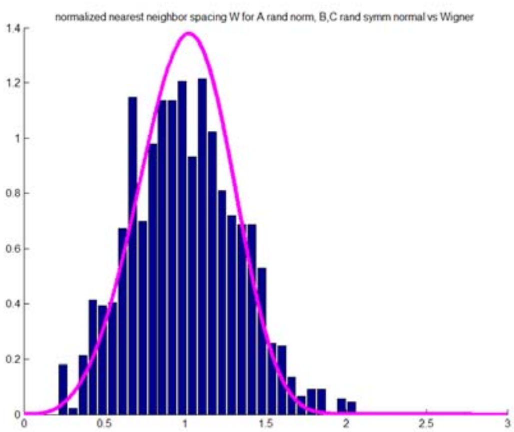
symmetry about real axis

Can view **W** as edge matrix for a weighted graph

We used Matlab command randn(1500) to get A,B,C matrices with random normally distributed entries mean 0 std dev 1

Nearest Neighbor Spacings vs Wigner surmise of Ginibre

$p_w(s) = 4\,\Gamma(5/4)^4\, s^3 \exp(-\Gamma(5/4)^4 s^4)$



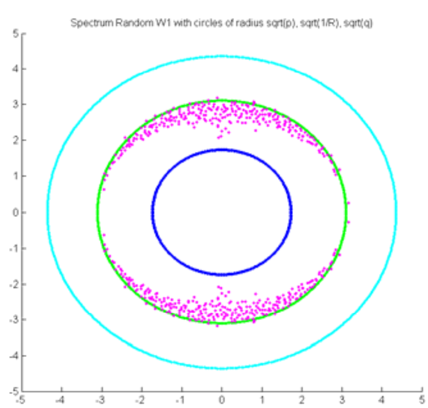
See P. Leboef, "Random matrices, random polynomials and Coulomb systems," ArXiv, Nov. 15, 1999.

Mehta, *Random Matrices*, Chapter 15

Note that we have normalized the histogram to have area 1.

The spacings are also normalized to have mean 1.

Matlab Experiments with Eigenvalues of "Random" W_1 matrix of an Irregular Graph - Reciprocals of Poles of Zeta



Circles have radii

$\sqrt{p}$ blue

$1/\sqrt{R}$ green

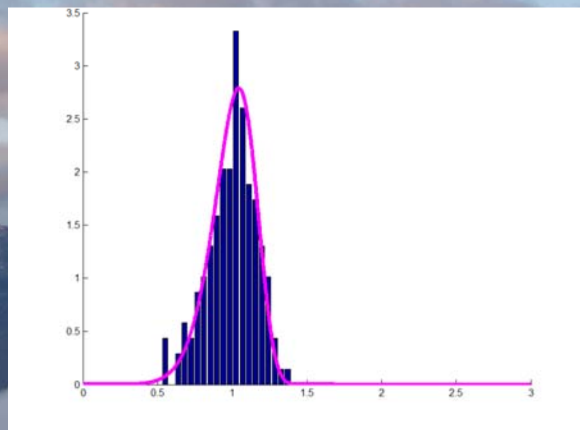
$\sqrt{q}$ turquoise

RH approximately true region 2 dimensional but not even an annulus

Looks very similar to the regions obtained for random covers of a small base graph.

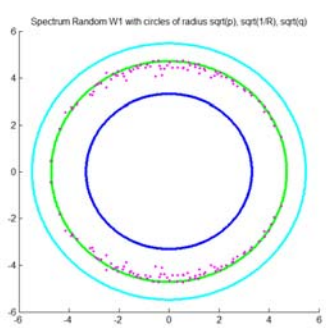
probability of edge ≅ 0.0358

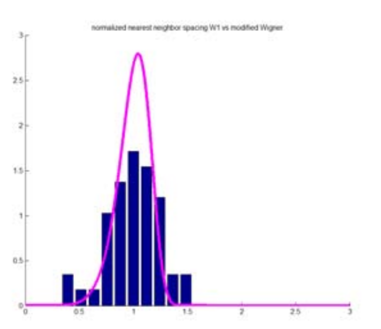
The corresponding nearest neighbor spacings for the preceding graph vs the Wigner surmise



Matlab Experiment Random Graph with Higher Probability of an Edge Between Vertices

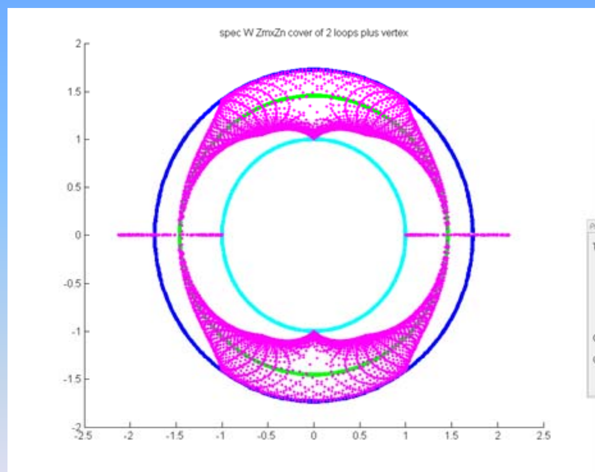
(Edge Probability ≈ 0.2848)

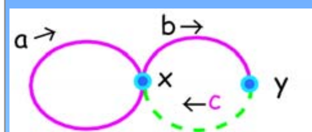




RH ≈ true

Spectrum W_1 for a $\mathbb{Z}_{61}\times\mathbb{Z}_{65}$-Cover of 2 Loops + Extra Vertex are pink dots

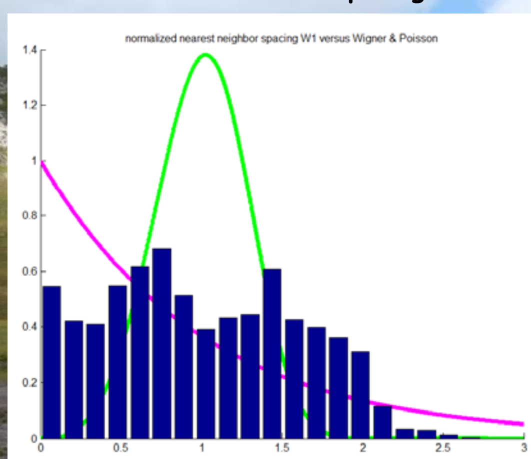




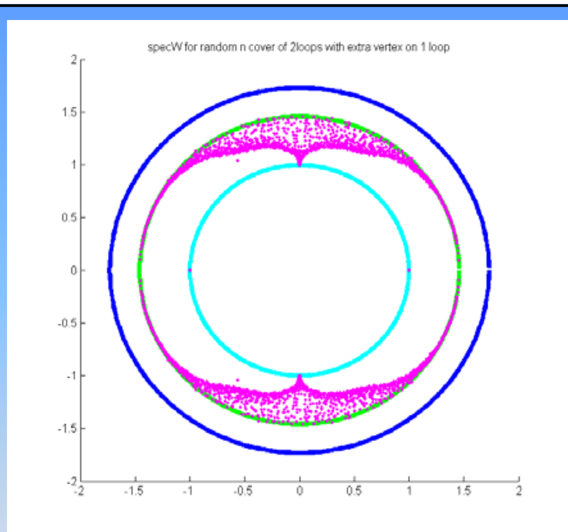
Circles Centers (0,0); Radii: $3^{-1/2}$, $R^{1/2}$, 1; $R \cong .47$

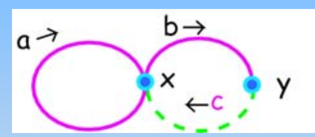
RH very False: Lots of Pink outside green circle

Level Spacing for the Previous W_1 Matrix

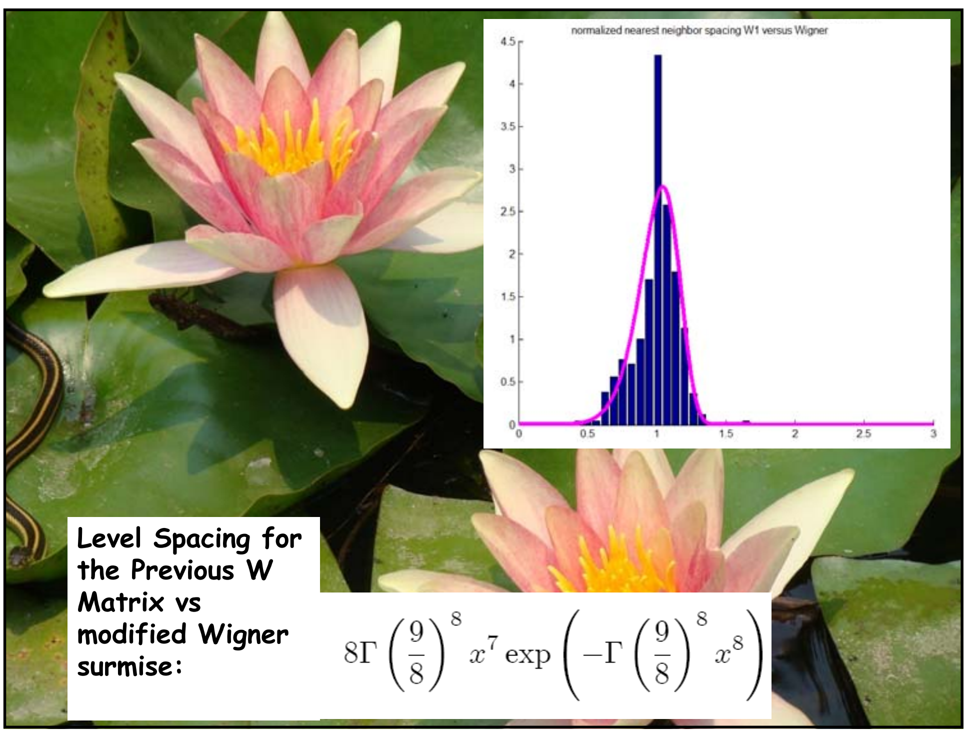


Comparing W_1 spacings for this abelian cover with the random cover following looks like the dichotomy between spacings of eigenvalues of the Laplacian for arithmetic vs non-arithmetic groups.





Spec W for random 701 cover of 2 loops plus vertex graph in picture. The pink dots are at Spectrum W. Circles have radii $\sqrt{q}$, $1/\sqrt{R}$, $\sqrt{p}$, with $q=3$, $p=1$, $R \cong .4694$. RH approximately True.



Level Spacing for the Previous W Matrix vs modified Wigner surmise:

$$8\Gamma\left(\frac{9}{8}\right)^{8} x^{7} \exp\left(-\Gamma\left(\frac{9}{8}\right)^{8} x^{8}\right)$$

References: 3 papers with Harold Stark in *Advances in Math.*

- Papers with Matthew Horton & Harold Stark on my website www.math.ucsd.edu/~aterras/
- For work on directed graphs, see Matthew Horton, Ihara zeta functions of digraphs, *Linear Algebra and its Applications,* 425 (2007) 130–142.
- work of Angel, Friedman and Hoory giving analog of Alon conjecture for irregular graphs, implying our Riemann Hypothesis (see Joel Friedman's website: www.math.ubc.ca/~jf)