Power Series and ODEs (5.1 & 5.2)

Example 1. Consider **$y' = ry$, $y(0) = c$**, where r and c are constants.

Plug $y(x) = \sum_{n=0}^{\infty} a_n x^n$. Then **$a_0 = c$**. Now differentiate.

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{m=0}^{\infty} (m+1) a_{m+1} x^m .$$

Here we set m = n-1.

So the ODE **$y' = ry$** implies $r\sum_{m=0}^{\infty} a_m x^m = \sum_{m=0}^{\infty} (m+1) a_{m+1} x^m .$

Since two power series are only equal if the corresponding coefficients are equal, we see that

$$ra_m = (m+1)a_{m+1}.$$

Thus we obtain the **recurrence relation:** **$a_{m+1} = ra_m /(m+1)$.**

The initial condition gave us $a_0 = c$. Then the recurrence relation says that

$$a_1 = r\frac{c}{1},\ a_2 = r\frac{a_1}{2} = r^2\frac{c}{2},\ \ a_3 = r\frac{a_2}{3} = r^3\frac{c}{6}.$$

This leads to the **formula for the mth coefficient** $a_m = c\frac{r^m}{m!}.$

(The proof of this formula requires mathematical induction. See Math. 109.)

Thus the power series is:

$$y(x) = c\sum_{n=0}^{\infty} \frac{(rx)^n}{n!} = ce^{rx}.$$

This should be no surprise, as **$y' = ry$, $y(0) = c$ was the 1st ode we solved.**

Example 2. **$y'' - y = 0$.**

Now there are 2 fundamental solutions and we should be able to find them using power series.

$$y(x)= \sum_{n=0}^{\infty} a_n x^n. \qquad y'(x)= \sum_{n=1}^{\infty} a_n n x^{n-1}.$$

$$y''(x) = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} = \sum_{m=0}^{\infty} a_{m+2}(m+2)(m+1) x^m.$$

For the last equality, we set n-2=m.

So the ODE implies $$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n.$$

This yields the recurrence relation $$a_n = (n+2)(n+1)a_{n+2} \text{ and } a_{n+2} = \frac{a_n}{(n+2)(n+1)}.$$

In particular, $$a_2 = \frac{a_0}{2}, \quad a_4 = \frac{a_2}{4\cdot 3} = \frac{a_0}{4\cdot 3\cdot 2}$$

So $a_0 = y(0)$ determines the even coefficients a_{2n} and $a_1 = y\,'(0)$ determines the odd coefficients a_{2n+1}.

We find that $$a_{2m} = \frac{a_0}{(2m)!}.$$

Similarly we find the odd coefficients and

$$a_3 = \frac{a_1}{3\times 2}, \quad a_5 = \frac{a_3}{5\times 4} = \frac{a_1}{5\times 4\times 3\times 2}.$$

This leads to $$a_{2m+1} = \frac{a_0}{(2m+1)!}.$$

It follows that $$y(x) = a_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} + a_1 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}.$$

You may recognize that the first power series is that for cosh x and the second is that for sinhx. Thus our solution is a linear combination of hyperbolic cosine and sine:

$$y(x) = a_0 \cosh x + a_1 \sinh x.$$

Example 3. Power Series Centered at Other Points that the Origin. Consider

$$xy'' + y' + xy = 0 \quad \text{with center at } 1.$$

(This is #8 in Section 5.2.)

So we plug $$y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n. \qquad y'(x) = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1}.$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2}.$$

Before plugging into the ODE, $xy'' + y' + xy = 0$, recall that $x = 1 + (x-1)$.

From **$xy'' + y' + xy = 0$, we get**

$$\begin{aligned}
0 &= x\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} + \sum_{n=1}^{\infty} na_n(x-1)^{n-1} + x\sum_{n=0}^{\infty} a_n(x-1)^n \\
&= (1+(x-1))\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} + \sum_{n=1}^{\infty} na_n(x-1)^{n-1} + (1+(x-1))\sum_{n=0}^{\infty} a_n(x-1)^n \\
&= \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-1} \\
&\quad + \sum_{n=1}^{\infty} na_n(x-1)^{n-1} + \sum_{n=0}^{\infty} a_n(x-1)^n + \sum_{n=1}^{\infty} a_n(x-1)^{n+1} \\
&= \sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2}(x-1)^m + \sum_{n=1}^{\infty} n^2a_n(x-1)^{n-1} + \sum_{n=0}^{\infty} a_n(x-1)^n + \sum_{k=2}^{\infty} a_{k-1}(x-1)^k
\end{aligned}$$

At the beginning we find one of our series is missing the constant term.

The coefficient of $(x-1)^0$ is $2a_2 + a_1 + a_0 = 0$ which implies that $a_2 = -\frac{a_1 + a_0}{2}$

The coefficient of $(x-1)$ is $6a_3 + 4a_2 + a_1 + a_0 = 0$

This says that $a_3 = -\frac{4a_2 + a_1 + a_0}{6}$

For general m

$$a_{m+2} = -\frac{(m+1)^2 a_{m+1} + a_m + a_{m-1}}{(m+2)(m+1)}$$

To find the fundamental solutions, we 1st assume $a_0=1$ and $a_1=0$ to get y_1 and then assume $a_1=0$ and $a_0=1$ to get y_2.

Case 1: $a_0=1$ and $a_1=0$

$$a_2 = -\frac{1}{2} \qquad a_3 = -\frac{-2+1}{6} = \frac{1}{6}$$

$$a_4 = -\frac{(2+1)^2 a_{2+1} + a_2 + a_1}{(2+2)(2+1)} = \frac{-1}{12}$$

$$y = 1 - \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 - \frac{1}{12}(x-1)^4 + \cdots$$

Case 2: $a_0=0$ and $a_1=1$

$$a_2 = -\frac{a_1 + a_0}{2} = -\frac{1}{2}, \quad a_3 = -\frac{-2+1}{6} = \frac{1}{6}$$

$$a_4 = -\frac{(2+1)^2 a_{2+1} + a_2 + a_1}{(2+2)(2+1)} = -\frac{1}{6}$$

$$y = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 - \frac{1}{6}(x-1)^4 + \cdots$$