Power Series and ODEs (5.1 & 5.2)

Example 1. Consider \( y' = ry, \ y(0) = c \), where \( r \) and \( c \) are constants.

Plug \( y(x) = \sum_{n=0}^{\infty} a_n x^n \). Then \( a_0 = c \). Now differentiate.

\[
y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{m=0}^{\infty} (m+1)a_{m+1} x^m.
\]

Here we set \( m = n-1 \).

So the ODE \( y' = ry \) implies \( r \sum_{m=0}^{\infty} a_m x^m = \sum_{m=0}^{\infty} (m+1)a_{m+1} x^m \).

Since two power series are only equal if the corresponding coefficients are equal, we see that \( r a_m = (m+1)a_{m+1} \).

Thus we obtain the recurrence relation: \( a_{m+1} = \frac{ra_m}{m+1} \).

The initial condition gave us \( a_0 = c \). Then the recurrence relation says that

\[
a_1 = r \frac{c}{1}, \quad a_2 = r \frac{a_1}{2} = r^2 \frac{c}{2}, \quad a_3 = r \frac{a_2}{3} = r^3 \frac{c}{6}.
\]

This leads to the formula for the \( m \)th coefficient \( a_m = \frac{r^m c}{m!} \).

(The proof of this formula requires mathematical induction. See Math. 109.)

Thus the power series is:

\[
y(x) = c \sum_{n=0}^{\infty} \frac{(rx)^n}{n!} = ce^{rx}.
\]

This should be no surprise, as \( y' = ry, \ y(0) = c \) was the 1st ode we solved.

Example 2. \( y'' - y = 0 \).

Now there are 2 fundamental solutions and we should be able to find them using power series.

\[
y(x) = \sum_{n=0}^{\infty} a_n x^n, \quad y'(x) = \sum_{n=1}^{\infty} a_n n x^{n-1}.
\]

\[
y''(x) = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} = \sum_{m=0}^{\infty} a_{m+2} (m+2)(m+1) x^m.
\]

For the last equality, we set \( n-2 = m \).
So the ODE implies
\[ \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_{n+2} (n+2)(n+1) x^n. \]

This yields the recurrence relation
\[
\begin{align*}
a_n &= (n+2)(n+1)a_{n+2} \quad \text{and} \\
a_{n+2} &= \frac{a_n}{(n+2)(n+1)}.
\end{align*}
\]

In particular,
\[
\begin{align*}
a_2 &= \frac{a_0}{2}, \\
a_4 &= \frac{a_2}{4 \cdot 3} = \frac{a_0}{4 \cdot 3 \cdot 2}.
\end{align*}
\]

So \(a_0 = y(0)\) determines the even coefficients \(a_{2n}\) and \(a_1 = y'(0)\) determines the odd coefficients \(a_{2n+1}\).

We find that
\[
a_{2m} = \frac{a_0}{(2m)!}.
\]

Similarly we find the odd coefficients and
\[
\begin{align*}
a_3 &= \frac{a_1}{3 \cdot 2}, \\
a_5 &= \frac{a_3}{5 \cdot 4} = \frac{a_1}{5 \cdot 4 \cdot 3 \cdot 2}.
\end{align*}
\]

This leads to
\[
a_{2m+1} = \frac{a_0}{(2m+1)!}.
\]

It follows that
\[
y(x) = a_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} + a_1 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}.
\]

You may recognize that the first power series is that for \(\cosh x\) and the second is that for \(\sinh x\).

Thus our solution is a linear combination of hyperbolic cosine and sine:
\[
y(x) = a_0 \cosh x + a_1 \sinh x.
\]

Example 3. Power Series Centered at Other Points than the Origin. Consider
\[
xy'' + y' + xy = 0 \quad \text{with center at } 1.
\]
(This is \#8 in Section 5.2.)

So we plug
\[
\begin{align*}
y(x) &= \sum_{n=0}^{\infty} a_n (x-1)^n, \\
y'(x) &= \sum_{n=1}^{\infty} na_n (x-1)^{n-1}.
\end{align*}
\]
$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2}.$$ 

Before plugging into the ODE, \( xy'' + y' + xy = 0 \), recall that \( x = 1 + (x-1) \).

From \( xy'' + y' + xy = 0 \), we get

\[
0 = x \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} + \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} + x \sum_{n=0}^{\infty} a_n (x-1)^{n}
\]

\[
= (1 + (x-1)) \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} + \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} + (1 + (x-1)) \sum_{n=0}^{\infty} a_n (x-1)^{n}
\]

\[
= \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} + \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-1}
\]

\[
+ \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} + \sum_{n=0}^{\infty} a_n (x-1)^{n} + \sum_{n=1}^{\infty} a_n (x-1)^{n+1}
\]

\[
= \sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} (x-1)^m + \sum_{n=1}^{\infty} n^2 a_n (x-1)^{n-1} + \sum_{n=0}^{\infty} a_n (x-1)^n + \sum_{k=2}^{\infty} a_{k-1} (x-1)^k
\]

At the beginning we find one of our series is missing the constant term.

The coefficient of \((x-1)^0\) is \(2a_2 + a_1 + a_0 = 0\) which implies that \(a_2 = -\frac{a_1 + a_0}{2}\).

The coefficient of \((x-1)^1\) is \(6a_3 + 4a_2 + a_1 + a_0 = 0\).

This says that \(a_3 = -\frac{4a_2 + a_1 + a_0}{6}\).

For general \(m\)

\[
a_{m+2} = -\frac{(m+1)^2 a_{m+1} + a_m + a_{m-1}}{(m+2)(m+1)}
\]

To find the fundamental solutions, we 1st assume \(a_0 = 1\) and \(a_1 = 0\) to get \(y_1\) and then assume \(a_1 = 0\) and \(a_0 = 1\) to get \(y_2\).

Case 1: \(a_0 = 1\) and \(a_1 = 0\)

\[
a_2 = -\frac{1}{2}, \quad a_3 = -\frac{2 + 1}{6} = \frac{1}{6}
\]
\[
a_4 = -\frac{(2+1)^2 a_{2+1} + a_2 + a_1}{(2+2)(2+1)} = \frac{-1}{12}
\]

\[
y = 1 - \frac{1}{2} (x-1)^2 + \frac{1}{6} (x-1)^3 - \frac{1}{12} (x-1)^4 + \cdots
\]

**Case 2:** \(a_0 = 0\) and \(a_1 = 1\)

\[
a_2 = -\frac{a_1 + a_0}{2} = -\frac{1}{2}, \quad a_3 = -\frac{2+1}{6} = \frac{1}{6}
\]

\[
a_4 = -\frac{(2+1)^2 a_{2+1} + a_2 + a_1}{(2+2)(2+1)} = \frac{-1}{6}
\]

\[
y = (x-1) - \frac{1}{2} (x-1)^2 + \frac{1}{6} (x-1)^3 - \frac{1}{6} (x-1)^4 + \cdots
\]