

What are zeta functions of graphs and what are they good for?

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ABSTRACT. We discuss zeta functions of finite irregular undirected connected graphs (which may be weighted) and apply them to obtain, for example an analog of the prime number theorem for cycles in graphs. We consider 3 types of zeta functions of graphs: vertex, edge, and path. Analogs of the Riemann hypothesis are also introduced.

1. Introduction

First let us tell some zeta stories.

The **Riemann zeta function** is defined for $\operatorname{Re}(s) > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p=\text{prime}} (1 - p^{-s})^{-1}.$$

In 1859 Riemann extended this function to all complex numbers s as a meromorphic function with its only pole (a simple one) at $s = 1$. He also showed that there is a functional equation relating $\zeta(s)$ and $\zeta(1-s)$. The functional equation involves the gamma function and says $\Lambda(s) = \pi^{-s/2} \Gamma(\frac{s}{2}) \zeta(s) = \Lambda(1-s)$.

Riemann's main interest in zeta was as a means to prove **the prime number theorem** (proved by Hadamard and de la Vallée Poussin in 1896) which says

$$\#\{p \mid p = \text{prime}, p \leq x\} \sim \frac{x}{\log x}, \text{ as } x \rightarrow \infty.$$

The Hadamard product for $s(s-1)\Lambda(s)$ gives a relation between the primes and the zeros of zeta. There are always zeros at the negative even integers (the trivial zeros). The still unproved Riemann hypothesis, which we will call **the RH**, says the non-trivial zeros of $\zeta(s)$ are on the line $\operatorname{Re}(s) = \frac{1}{2}$. It has been checked for a ridiculously large number of zeros. The RH has implications for the error term in the prime number theorem. A reference containing a translation of Riemann's paper is Edwards [7]. Recently it has been noticed that (assuming the RH) the spacings of the imaginary parts of the non-trivial zeros of $\zeta(s)$ behave like the eigenvalues of random Hermitian matrices. This relates zeta and quantum chaos. Katz and Sarnak [16] call this the Montgomery-Odlyzko **law** (another name is **GUE** for Gaussian Unitary Ensemble) although the law does not appear to have found a proof.

Many other kinds of zeta functions have been investigated since Riemann. In number theory there is the Dedekind zeta function of an algebraic number field K , such as $\mathbb{Q}(\sqrt{3})$, for example. This zeta is an infinite product over prime ideals \mathfrak{p} in O_K , the ring of algebraic integers of K . The terms in the product are $(1 - N\mathfrak{p}^{-s})^{-1}$, where $N\mathfrak{p} = \#(O_K/\mathfrak{p})$. Riemann's work can be extended to this function and it can be used to prove the prime ideal theorem. The RH is unproved but conjectured to be true for the Dedekind zeta function, except for some modifications concerning a possible zero near 1 on the real axis called a "Siegel zero." A reference for this zeta is Lang [19]. There are also function field zeta functions where K is replaced by a finite algebraic extension of $\mathbb{F}_q(x)$, the rational functions of one variable over a finite field \mathbb{F}_q . The RH has been proved for this zeta which is a rational function of $u = q^{-s}$. See Rosen [24]. Moreover, the Montgomery-Odlyzko behavior has been proved for function fields of generic curves as q and the genus approach infinity. See Katz and Sarnak [16], [17].

Selberg defined a zeta function in 1956 (see [26]) that is associated to a compact Riemannian manifold $M = \Gamma \backslash H$, where H is the Poincaré upper half plane with the metric $ds^2 = y^{-2}(dx^2 + dy^2)$. Here Γ is a discrete group of real fractional linear transformations of positive determinant. The "primes" in this case are primitive closed geodesics P in M of Poincaré length $\nu(P)$. The **Selberg zeta function** of M is

$$Z(s) = \prod_{[P]} \prod_{j=0}^{\infty} \left(1 - e^{-(s+j)\nu(P)}\right).$$

In fact, $Z(s+1)/Z(s)$ is more like the Riemann zeta function. References are Elstrodt [8], Hejhal [11], and Terras [32]. The Selberg trace formula gives a relation between the primes of M and the eigenvalues of the Poincaré Laplacian $\Delta = y^s \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$ on M . For compact M , the Selberg zeta function satisfies the analog of the RH.

Finally there is the Ruelle zeta function of a dynamical system. See Ruelle [25] or Bedford, Keane, and Series [2].

The graph theory zetas first appeared in work of Ihara on p-adic groups in the 1960s. Soon the connection with graphs was found and many papers appeared. The main authors in the 1980s and 90s were Sunada, Hashimoto [10], and Bass [1]. Other references are Venkov and Nikitin [33] and Northshield's paper in the volume of Hejhal et al [12]. See [31] for more history and references. The main properties of the Riemann zeta function have graph theory analogs, at least for regular graphs. For irregular graphs there is no known functional equation and it is difficult to formulate the Riemann hypothesis, but we will try in Section 2. In Sections 3 and 4 we will consider the multivariable zeta functions known as edge and path zeta functions of graphs. We will show how to specialize the path zeta to the edge zeta and the edge zeta to the original Ihara (vertex) zeta.

Here we attempt to survey some of the basic theory as presented in Stark and Terras [28], [29], [30]. We include the graph theory prime number theorem and our version of Bass's proof of the determinant formula. We will omit the Artin L-functions of Galois graph coverings which are developed in [29], for want of space. We will show how to modify the definitions to obtain zeta functions of weighted graphs. We will show what happens to the edge zeta if you delete an edge (fission) and what happens to the path zeta if you collapse an edge to a point (fusion).

We do not consider zeta functions of infinite graphs in this paper. Such zeta functions are considered, for example, by Clair and Mokhtari-Sharghi [5] as well as Grigorchuk and Zuk [9].

2. Vertex Zeta Functions

First we need to find out what primes in graphs are.

We will always assume that our graphs X are finite, connected, rank ≥ 1 with no danglers (i.e., degree 1 vertices). Here “rank” means the rank of the fundamental group of the graph (given more explicitly in the next paragraph). Let us recall some of the definitions basic to Stark and Terras [28], [29], [30].

If X is any connected finite undirected graph with vertex set V and (undirected) edge set E , we **orient its edges arbitrarily and obtain $2|E|$ oriented edges labelled**

$$(2.1) \quad e_1, e_2, \dots, e_n, e_{n+1} = e_1^{-1}, \dots, e_{2n} = e_n^{-1}.$$

“**Primes**” $[P]$ in X are equivalence classes of closed backtrackless tailless primitive paths P . Write a path $C = a_1 a_2 \cdots a_s$, where a_j is an oriented edge of X . The **length** of C is $\nu(C) = s$. **Backtrackless** means that $a_{i+1} \neq a_i^{-1}$, for all i . **Tailless** means that $a_s \neq a_1^{-1}$. The **equivalence class** $[C]$ is the set

$$[C] = \{a_1 a_2 \cdots a_s, a_2 a_3 \cdots a_s a_1, \dots, a_s a_1 \cdots a_{s-1}\}.$$

$[P]$ is **primitive** means $P \neq D^m$, for any integer $m \geq 2$ and path D in X . Here r_X will denote the **rank** of the fundamental group of X . We have $r_X - 1 = |E| - |V|$. Then r_X is the number of edges deleted from X to form a spanning tree.

The **Ihara (vertex) zeta function** of X is defined at $u \in \mathbb{C}$, for $|u|$ sufficiently small, by

$$(2.2) \quad \zeta(u, X) = \zeta_V(u, X) = \prod_{[P]} \left(1 - u^{\nu(P)}\right)^{-1},$$

where $[P]$ runs over the primes of X . Here V is for vertex. In this section, we will drop V most of the time for the vertex zeta. In later sections we will define two other kinds of zeta associated to a graph .

The generalization of **Ihara’s formula** (proved by Hashimoto and Bass) says

$$(2.3) \quad \zeta(u, X)^{-1} = (1 - u^2)^{r_X - 1} \det(I - A_X u + Q_X u^2),$$

where r_X is the rank of the fundamental group of X , A_X is the adjacency matrix of X , Q_X is the diagonal matrix whose j th diagonal entry is $(-1 + \text{degree of } j\text{th vertex})$. We will give a version of Bass’s proof of Ihara’s formula in the next section. There are many other proofs. See Stark and Terras [28] for one such. In Terras [31] there is a proof for k -regular graphs using the Selberg trace formula on the k -regular tree.

DEFINITION 2.1. R_X is the **radius of the largest circle of convergence** of the Ihara zeta function.

When X is a $(q + 1)$ -regular graph, $R_X = 1/q$. We will say more about the size of R_X for irregular graphs later in this section. As with the Dedekind zeta function, $\zeta(u, X)$ has a meromorphic continuation to the entire complex u -plane, but now $\zeta(u, X)^{-1}$ is entire. Thus if we want to formulate a version of the RH=Riemann Hypothesis, we must consider the poles of $\zeta(u, X)$, not the zeroes.

In general, $\zeta(u, X)^{-1}$ is a polynomial of degree $2m$, where $m = |E|$, the number of edges of X .

DEFINITION 2.2. We define the **0,1 edge matrix** W_1 by orienting the m edges of X and labeling them as in formula (2.1). Then W_1 is the $2m \times 2m$ matrix with ij entry 1 if edge e_i feeds into e_j provided that $e_j \neq e_i^{-1}$, and ij entry 0 otherwise. By “ a feeds into b ,” we mean that the terminal vertex of edge a is the same as the initial vertex of edge b .

We will show in the next section (see Corollary 1) that

$$(2.4) \quad \zeta(u, X)^{-1} = \det(I - W_1 u).$$

From this we shall derive **Ihara’s formula (2.3)**.

DEFINITION 2.3. For a graph X with m edges which have weights given by a weighting function $L : E \rightarrow \mathbb{R}^+$, define the **weighted length** of a closed path $C = a_1 a_2 \cdots a_s$, where a_j is an oriented edge of X , by

$$\nu(C, L) = \nu_X(C, L) = \sum_{i=1}^s L(a_i).$$

DEFINITION 2.4. The **Ihara zeta function of a weighted (undirected) graph** for $|u|$ small and $u \notin (-\infty, 0)$ is

$$\zeta(u, X, L) = \prod_{[P]} \left(1 - u^{\nu(P, L)}\right)^{-1}.$$

Clearly when $L = 1$, meaning the function such that $L(e) = 1$ for all edges e in X , we have $\zeta(u, X, 1) = \zeta(u, X)$, our original Ihara zeta function. Then one would want to find a determinant formula for this weighted zeta function. We can find an Ihara determinant formula for integer weights by inflating the graph.

DEFINITION 2.5. Given a graph X with integer-valued weight function L , define the **inflated graph** X_L in which each edge e is replaced by an edge with $L(e) - 1$ new degree 2 vertices.

Then clearly $\nu_X(C, L) = \nu_{X_L}(C, 1)$, where the 1 means again that $1(e) = 1$, for all edges e . It follows that **for integer-valued weights L , we have the identity relating the weighted zeta and the ordinary Ihara zeta:**

$$\zeta(u, X, L) = \zeta(u, X_L),$$

and thus an Ihara determinant formula holds for this integer weighted graph zeta. It follows that $\zeta(u, X, L)^{-1}$ is a polynomial for integer valued weights L . For non-integer weights, we get a determinant formula for the general weighted Ihara zeta function in the next section.

Example. Suppose $Y = K_5$, the complete graph on 5 vertices. Let $L(e) = 5$ for each of the 10 edges of X . Then $X = Y_L$ is the graph on the left in Figure 1. One sees easily that $\zeta(u^5, K_5) = \zeta(u, X) = (1 - u^{10})^5 (1 - 3u^5) (1 - u^5) (1 + u^5 + 3u^{10})$.

Kotani and Sunada [18] show that, for a non-circuit graph X , if $q + 1$ is the maximum degree of X and $p + 1$ is the minimum degree of X , then every non-real pole u of $\zeta_X(u)$ satisfies the inequality

$$(2.5) \quad q^{-1/2} \leq |u| \leq p^{-1/2}.$$

Moreover, they show that every pole u of $\zeta_X(u)$ satisfies $R_X \leq |u| \leq 1$ and

$$(2.6) \quad q^{-1} \leq R_X \leq p^{-1}.$$

Next let us speak about Ramanujan graphs. These were defined for connected regular graphs by Lubotzky, Phillips, and Sarnak [22] and are of interest to computer scientists because the standard random walk on such graphs converges as fast as possible to uniform (at least in the non bipartite case) for a sequence of connected k -regular graphs such that the number of vertices approaches infinity. See Terras [31] and the references contained there. Other references for Ramanujan graphs are: Fan Chung [4], Giuliana Davidoff, Peter Sarnak and Alain Valette [6], Alexander Lubotzky [20], and the paper of W.-C. Winnie Li in the volume of Hejhal et al [12].

First define two constants associated to the graph X .

DEFINITION 2.6.

$$\begin{aligned} \rho_X &= \max \{ |\lambda| \mid \lambda \in \text{spectrum}(A_X) \}, \\ \rho'_X &= \max \{ |\lambda| \mid \lambda \in \text{spectrum}(A_X), |\lambda| \neq \rho_X \}. \end{aligned}$$

We will say that the **naive Ramanujan inequality** is

$$(2.7) \quad \rho'_X \leq 2\sqrt{\rho_X - 1}.$$

Lubotzky [21] has defined X to be **Ramanujan** if

$$(2.8) \quad \rho'_X \leq \sigma_X.$$

where σ_X is the **spectral radius of the adjacency operator on the universal covering tree of X** . Both inequalities (2.7) and (2.8) reduce to the usual definition of Ramanujan for connected regular graphs.

Hoory [13] has proved that if $\overline{d_X}$ denotes the **average degree of the vertices** of X , then

$$2\sqrt{\overline{d_X} - 1} \leq \sigma_X.$$

Thus one has a criterion for a graph X to be Ramanujan in Lubotzky's sense. It need only satisfy the **Hoory inequality**

$$(2.9) \quad \rho'_X \leq 2\sqrt{\overline{d_X} - 1}.$$

For a $(q+1)$ -regular graph the standard change of variables $u = q^{-s}$, $s \in \mathbb{C}$, turns $\zeta(u, X)$ into a Dirichlet series which is zero- and pole-free for $\text{Re}(s) > 1$ and has a first order pole at $s = 1$. In the new s -variable, there are functional equations relating the value of $f(s) = \zeta(q^{-s}, X)$ with $f(1-s)$ (see [28]). The RH (= Riemann hypothesis) says (e.g., see [28], p. 129) $\zeta_X(u)$ has no poles with $0 < \text{Re}(s) < 1$ unless $\text{Re}(s) = 1/2$. D. Newland [23] has done computer experiments showing that for the pseudo-random regular graphs produced by Mathematica the spacings of the imaginary parts of the poles of $f(s) = \zeta(q^{-s}, X)$ appear to be those of the eigenvalues of a random real symmetric matrix (GOE = Gaussian Orthogonal Ensemble) with a change of variables coming from the formula relating the poles of the Ihara zeta function and the eigenvalues of the adjacency matrix.

To develop the RH for irregular graphs, the natural change of variable is $u = R_X^s$ with R_X from Definition 2.1. All poles of $\zeta_X(u)$ are then located in the "critical strip", $0 \leq \text{Re}(s) \leq 1$ with poles at $s = 0$ ($u = 1$) and $s = 1$ ($u = R_X$). From this point of view, it is natural to say that the Riemann hypothesis for X should

require that $\zeta_X(u)$ has no poles in the open strip $1/2 < \operatorname{Re}(s) < 1$. This is the graph theory RH below. After looking at examples, it seems that one rarely sees an Ihara zeta satisfying this RH so we weaken it to the weak graph theory RH below. The examples below show that in general one cannot expect a functional equation relating $f(s) = \zeta(R_X^s, X)$ and $f(1-s)$.

(Graph theory RH) $\zeta_X(u)$ is pole free for

$$(2.10) \quad R_X < |u| < \sqrt{R_X},$$

(Weak graph theory RH) $\zeta_X(u)$ is pole free for

$$(2.11) \quad R_X < |u| < 1/\sqrt{q}.$$

Note that we have examples (such as Examples 1 and 4 below) for which $R_X > q^{-\frac{1}{2}}$ and in such cases the weak graph theory RH is true but vacuous. In [30] we give a longer discussion of the preceding 2 versions of the RH for graphs showing the connections with the versions for the Dedekind zeta function and the existence of Siegel zeros.

Sometimes number theorists state a modified GRH = Generalized Riemann Hypothesis for the Dedekind zeta function and this just ignores all possible real zeros while only requiring the non-real zeros to be on the line $\operatorname{Re}(s) = \frac{1}{2}$. The graph theory analog of the modified weak GRH would just ignore the real poles and require that there are no non-real poles of $\zeta_X(u)$ in $R_X < |u| < q^{-1/2}$. But this is true for all graphs by a theorem in [18] quoted above in (2.5): if μ is a pole of $\zeta_X(u)$ and $|\mu| < q^{-1/2}$ then μ is real!

One may ask about the relations between the constants $\rho_X, \overline{d_X}, R_X$. One can show that

$$(2.12) \quad \rho_X \geq \overline{d_X}.$$

This is easily seen using the fact that ρ_X is the maximum value of the Rayleigh quotient $\langle Af, f \rangle / \langle f, f \rangle$, while $\overline{d_X}$ is the value when f is the vector all of whose entries are 1. In all the examples to date we see that $\rho_X \geq 1 + \frac{1}{R_X} \geq \overline{d_X}$ but can only show that $\rho_X \geq \frac{2}{q} + \frac{1}{R_X}$.

Next we give some examples including answers to the questions: Do the spectra of the adjacency matrices satisfy the naive Ramanujan inequality (2.7) or the Hoory inequality (2.9)? Do the Ihara zeta functions for the graphs have the pole-free region (2.11) of the weak graph theory RH or the pole-free region (2.10) of the full graph theory RH?

Example 1. X = the graph obtained from K_5 by adding 4 vertices to each edge as shown on the left in Figure 1.

We thank Adam O’Neill for his help with this example. For the graph X , we find that $\rho' \approx 2.32771$ and $\{\rho, 1 + 1/R, \overline{d_X}\} \approx \{2.39138, 2.24573, 2.22222\}$. It follows that this graph satisfies the naive Ramanujan inequality (2.7) but not the Hoory inequality (2.9). The picture on the right in Figure 1 shows stars for the poles not equal to -1 of $\zeta(u, X)$. Here

$$\zeta(u, X)^{-1} = \zeta(u^5, K_5)^{-1} = (1 - u^{10})^5 (1 - 3u^5) (1 - u^5) (1 + u^5 + 3u^{10}).$$

The circles in the picture on the right in Figure 1 are centered at the origin with radii $\{q^{-\frac{1}{2}}, R, R^{\frac{1}{2}}, p^{-\frac{1}{2}}\} \approx \{0.57735, 0.802742, 0.895958, 1\}$. The zeta function satisfies the RH and thus the weak RH. However the weak RH is vacuous.

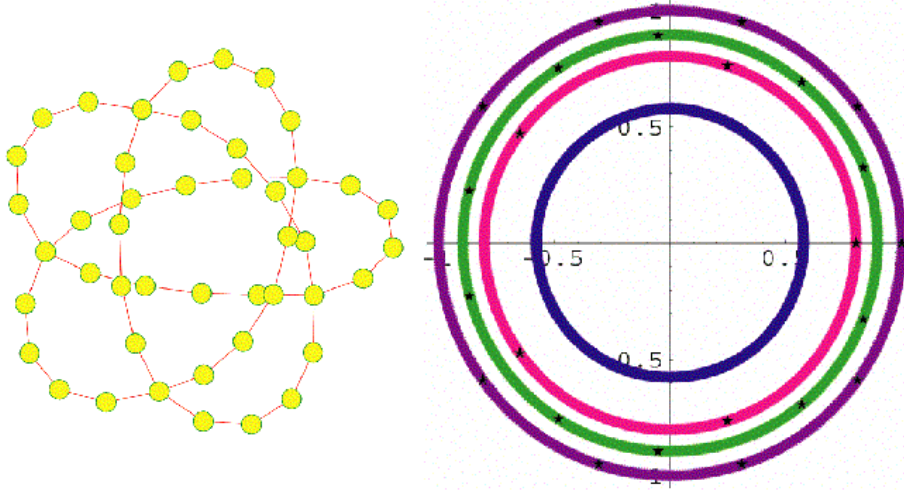


FIGURE 1. On the left is the graph $X = Y_5$ obtained by adding 4 vertices to each edge of $Y = K_5$, the complete graph on 5 vertices. On the right the poles ($\neq -1$) of the vertex Ihara zeta function of X are shown by stars. The circles have centers at the origin and radii $\{q^{-\frac{1}{2}}, R, R^{\frac{1}{2}}, p^{-\frac{1}{2}}\}$.

Example 2. Random graph with probability 1/2 of an edge.

See Figure 2 for the location of the poles not equal to ± 1 of the vertex Ihara zeta function of a random graph produced by Mathematica with the command `RandomGraph[100,1/2]`. This means there are 100 vertices and the probability of an edge between any 2 vertices is 1/2. The graph satisfies the Hoory inequality (2.9) and it is thus Ramanujan in Lubotzky's sense and satisfies the naive Ramanujan inequality (2.7). We find that $\rho' \approx 10.3357$ and $\{\rho, 1 + 1/R, \overline{d_X}\} \approx \{48.5972, 48.585, 48.\}$. The circles in Figure 2 are centered at the origin and have radii given by $\{R, q^{-\frac{1}{2}}, R^{\frac{1}{2}}, p^{-\frac{1}{2}}\} \approx \{0.021015, 0.129099, 0.144966, 0.174078\}$. The poles of the zeta function satisfy the weak RH but not the RH. See Skiena [27] for more information on the model that Mathematica uses to produce random graphs.

Example 3. Torus minus some edges.

From the torus graph T which is the product of a 3-cycle and a 10-cycle, we delete 7 edges to obtain a graph we will call N which is on the left in Figure 3. The spectrum of the adjacency matrix of N satisfies neither the Hoory inequality (2.9) nor the naive Ramanujan inequality (2.7). We find that $\{\rho, 1 + 1/R, d\} \approx \{3.60791, 3.58188, 3.53333\}$, and $\rho' \approx 3.39492$. The right hand side of Figure 3 shows the poles of the vertex zeta for N as stars. The circles have center at the origin and radii $\{R, q^{-\frac{1}{2}}, R^{\frac{1}{2}}, p^{-\frac{1}{2}}\} \approx \{0.387315, 0.57735, 0.622347, 0.707107\}$. The zeta poles satisfy neither the graph theory weak RH nor the RH.

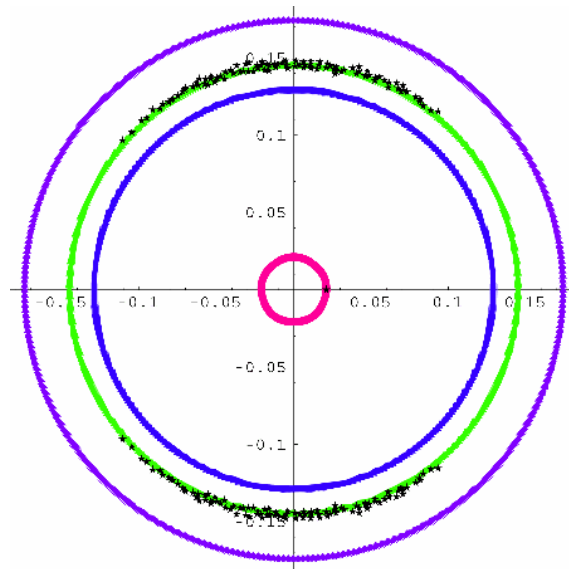


FIGURE 2. The poles ($\neq \pm 1$) of the vertex Ihara zeta function for a random graph produced by Mathematica with the command `RandomGraph[100, $\frac{1}{2}$]`. The circles have centers at the origin and radii $\{R, q^{-\frac{1}{2}}, R^{\frac{1}{2}}, p^{-\frac{1}{2}}\}$.

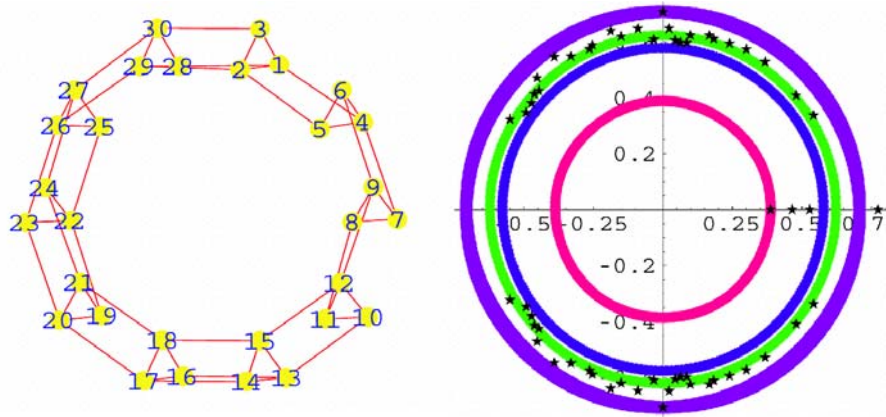


FIGURE 3. The graph N on the left results from deleting 7 edges from the product of a 3-cycle and a 10-cycle. In the picture on the right, stars indicate the poles ($\neq \pm 1$) of the vertex Ihara zeta function of N . The circles are centered at the origin with radii $\{R, q^{-1/2}, R^{1/2}, p^{-1/2}\}$.

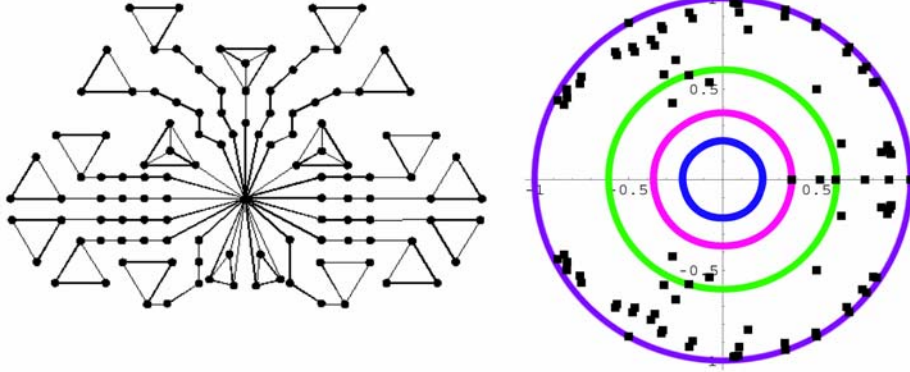


FIGURE 4. On the left is the e-graph E of Example 4 with the property that R^{-1} approximates e to 7 digits. On the right the poles ($\neq -1$) of the vertex Ihara zeta function of E are indicated by boxes. The circles have centers at the origin and radii $\{q^{-\frac{1}{2}}, R, R^{\frac{1}{2}}, p^{-\frac{1}{2}}\}$.

Example 4. The e-graph E found by M. Horton [15] has the property that $1/R \approx 2.7182819$. It is pictured on the left in Figure 4.

We find that $\{\rho, 1 + 1/R, d\} \approx \{5.26243, 3.71828, 2.47863\}$. The spectrum of the adjacency matrix of E satisfies neither the Hoory inequality (2.9) nor the naive Ramanujan inequality (2.7). It is open whether it is Ramanujan in Lubotzky's sense. The right hand side of Figure 4 shows the poles ($\neq -1$) of the vertex zeta function of E as boxes. The circles have center at the origin and radii $\{q^{-\frac{1}{2}}, R, R^{\frac{1}{2}}, p^{-\frac{1}{2}}\} \approx \{0.213201, 0.367879, 0.606531, 1\}$. The weak RH is vacuous and thus true, while the RH is false.

Next we intend to prove the graph theory prime number theorem. First we need some definitions.

DEFINITION 2.7. The **closed path counting function** N_m is the number of closed paths C of length m in X without backtracking or tails.

DEFINITION 2.8. The **prime counting function** is

$$\pi(n) = \#\{[C] = \text{prime in } X \text{ with length of } C \text{ equal to } n\}.$$

Note that in $\pi(n)$ we count equivalence classes of paths and not paths.

DEFINITION 2.9. The **g.c.d. of the prime path lengths** in X is

$$\Delta_X = \text{g.c.d.}\{\nu(P) \mid [P] = \text{a prime in } X\}.$$

THEOREM 2.10. (Graph Prime Number Theorem) Suppose that R_X is as in Definition 2.1. Then for a connected graph X with $\Delta_X = 1$ we have

$$\pi(m) \sim \frac{R_X^{-m}}{m}, \quad \text{as } m \rightarrow \infty.$$

If $\Delta_X > 1$, then $\pi(m) = 0$ unless Δ_X divides m . If Δ_X divides m , we have

$$\pi(m) \sim \Delta_X \frac{R_X^{-m}}{m}, \quad \text{as } m \rightarrow \infty.$$

PROOF. Recall formula (2.2) defining the Ihara zeta function. Then

$$\begin{aligned} u \frac{d}{du} \log \zeta(u, X) &= -u \frac{d}{du} \sum_{[P]} \log(1 - u^{\nu(P)}) \\ &= \sum_{[P]} \sum_{j \geq 1} \frac{1}{j} u \frac{d}{du} u^{\nu(P)j} = \sum_{[P]} \sum_{j \geq 1} \frac{\nu(P)j}{j} u^{\nu(P)j}. \end{aligned}$$

The outer sum is over equivalence classes $[P]$ of prime closed paths with no backtracking or tails. Sum over all primes $[P]$ with $\nu(P) = d$. Now there are d elements in $[P]$. So we can drop the $[]$ and obtain a sum over all closed paths $C = P^j$ without backtracking or tails. This gives

$$(2.13) \quad u \frac{d}{du} \log \zeta(u, X) = \sum_{m \geq 1} N_m u^m.$$

Next observe that the defining formula for the Ihara zeta function can be written as

$$\zeta(u, X) = \prod_{n \geq 1} (1 - u^n)^{\pi(n)}.$$

Then

$$u \frac{d}{du} \log \zeta(u, X) = \sum_{n \geq 1} \frac{n\pi(n)u^n}{1 - u^n} = \sum_{m \geq 1} \sum_{d|m} d\pi(d)u^m.$$

Here the inner sum is over all positive divisors of m . Thus we obtain the following relation between the functions on \mathbb{Z}^+ which we have called N_m and $\pi(n)$.

$$N_m = \sum_{d|m} d\pi(d).$$

This sort of relation occurs frequently in number theory and combinatorics. It is inverted using the **Möbius function** $\mu(n)$ defined by

$$\mu(n) = \begin{cases} 1 & n = 1 \\ (-1)^r & n = p_1 \cdots p_r, \text{ for distinct primes } p_i \\ 0 & \text{otherwise.} \end{cases}$$

Then by the Möbius inversion formula

$$(2.14) \quad \pi(m) = \frac{1}{m} \sum_{d|m} \mu\left(\frac{m}{d}\right) N_d.$$

Next we look at formula (2.4) and apply our differential operator after using the Schur upper triangularization of W_1 (see Horn and Johnson [14]). This gives

$$u \frac{d}{du} \log \zeta(u, X) = - \sum_{\lambda \in \text{Spec}(W_1)} \log(1 - \lambda u) = \sum_{\lambda \in \text{Spec}(W_1)} \sum_{n \geq 1} \frac{1}{n} (\lambda u)^n.$$

It follows that

$$(2.15) \quad N_m = \sum_{\lambda \in \text{Spec}(W_1)} \frac{\lambda^m}{m}.$$

The dominant terms in this last sum are those coming from $\lambda \in \text{Spec}(W_1)$ such that $|\lambda| = R^{-1}$. By Kotani and Sunada [18], Theorem 1.4, which is proved using the Perron-Frobenius theorem (see [14]), the largest absolute value of an eigenvalue λ occurs Δ_X times with these eigenvalues having the form $e^{2\pi ia/\Delta_X} R$, where $a = 1, \dots, \Delta_X$. Using the orthogonality relations for exponential sums (see [31]) which are basic to the theory of the finite Fourier transform, we see that

$$(2.16) \quad \sum_{|\lambda| \text{ maximal}} \frac{\lambda^n}{n} = \frac{R^{-n}}{n} \sum_{a=1}^{\Delta_X} e^{\frac{2\pi i a n}{\Delta_X}} = \frac{R^{-n}}{n} \begin{cases} 0, & \Delta_X \text{ does not divide } n \\ \Delta_X, & \Delta_X \text{ divides } n. \end{cases}$$

The graph prime number theorem follows from formulas (2.14), (2.15), and (2.16). □

The vertex Ihara zeta function can be used to determine the rank of the fundamental group, for it is the order of the pole of the Ihara zeta function at $u = 1$. The complexity κ_X of a graph is defined to be the number of spanning trees in X . One can use the matrix-tree theorem (see Biggs [3]) to prove that

$$\left(\frac{1}{\zeta_V(u, X)} \right)^{(r)} (1) = r!(-1)^{r+1} 2^r (r-1) \kappa_X.$$

Here the superscript (r) means the rth derivative. This result is an exercise on the last page of Terras [31], where some hints are given. It is an analog of the formula for the Dedekind zeta function of a number field at 0 - a formula involving the class number and the regulator of the number field. See Lang [19].

3. Edge Zeta Functions

Here we will give our version of the Bass proof of Ihara's determinant formula. This implies that there is a determinant formula for the vertex zeta function of weighted graphs even if the weights are not integers. In an example we will discuss what deleting an edge of a graph (fission) does to the edge zeta function.

DEFINITION 3.1. The **edge matrix** W for graph X is a $2m \times 2m$ matrix with a, b entry corresponding to the oriented edges a and b . This a, b entry is the complex variable w_{ab} if edge a feeds into edge b and $b \neq a^{-1}$ and the a, b entry is 0 otherwise.

DEFINITION 3.2. Given a path C in X , which is written as a product of oriented edges $C = a_1 a_2 \cdots a_s$, the **edge norm** of C is

$$N_E(C) = w_{a_1 a_2} w_{a_2 a_3} \cdots w_{a_{s-1} a_s} w_{a_s a_1}.$$

The **edge Ihara zeta function** is

$$\zeta_E(W, X) = \prod_{[P]} (1 - N_E(P))^{-1},$$

where the product is over primes in X .

Clearly if you set all non-zero variables in W equal to u , the **edge zeta function specializes to the vertex zeta function**; i.e.,

$$(3.1) \quad \zeta_E(W, X)|_{0 \neq w_{ab}=u} = \zeta_V(u, X).$$

If X is a weighted graph with weight function L , and you specialize the non-zero variables $w_{ab} = u^{(L(a)+L(b))/2}$, you get the weighted Ihara zeta function. Or you could specialize $w_{ab} = u^{L(a)}$.

If you cut an edge of a graph, you can compute the edge zeta for the new graph with one less edge by setting all variables equal to 0 if the cut edge or its inverse appear in a subscript.

The edge zeta again has a determinant formula and is the reciprocal of a polynomial in the w_{ab} variables. This is the following theorem whose proof should be compared with the Bowen Lanford proof of a similar theorem for dynamical zeta functions. See Ruelle [25] or Bedford, Keane, and Series [2].

THEOREM 3.3. (*Determinant Formula for the Edge Zeta*).

$$\zeta_E(W, X) = \det(I - W)^{-1}.$$

PROOF. (Sketch) First note that

$$-\log \zeta_E(W, X) = \sum_{[P]} \sum_{j \geq 1} \frac{1}{j} N_E(P)^j.$$

Let $L = \sum_{i,j} w_{ij} \frac{\partial}{\partial w_{ij}}$ be the Euler operator. Then, since there are $\nu(P)$ elements in $[P]$, we have

$$-L \log \zeta_E(W, X) = \sum_{\substack{m \geq 1 \\ j \geq 1}} \frac{1}{jm} \sum_{\substack{P \\ \nu(P)=m}} L(N_E(P)^j).$$

Now $\nu(P) = m$ implies $L(N_E(P)^j) = jmN_E(P)^j$. It follows that

$$-L \log \zeta_E(W, X) = \sum_C N_E(C).$$

Here the sum is the path C need not be a prime path, but it is still closed without backtracking or tail. Finally we see that

$$-L \log \zeta_E(W, X) = \sum_{m \geq 1} \text{Tr}(W^m).$$

Next you need to do an exercise in matrix calculus and show that the right hand side of the preceding formula is $L \log \det(I - W)^{-1}$. This proves $L(\log(\text{determinant formula}))$. To finish the proof, observe that both sides are 1 when all the w_{ab} are 0. \square

By formula (3.1), we have the following Corollary, since specializing all the non-zero variables in W to be u , yields the matrix uW_1 .

COROLLARY 1. $\zeta_V(u, X) = \det(I - uW_1)^{-1}$.

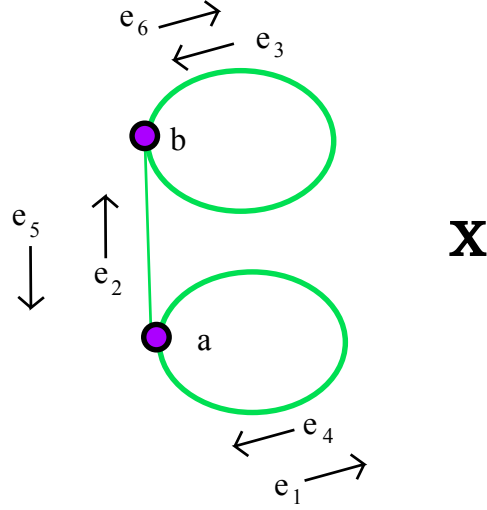


FIGURE 5. The dumbbell graph

Example. Dumbbell Graph.

Figure 5 shows the labeled picture of the dumbbell graph X . For this graph we find that $\zeta_E(W, X)^{-1} =$

$$\det \begin{pmatrix} w_{11} - 1 & w_{12} & 0 & 0 & 0 & 0 \\ 0 & -1 & w_{23} & 0 & 0 & w_{26} \\ 0 & 0 & w_{33} - 1 & 0 & w_{35} & 0 \\ 0 & w_{42} & 0 & w_{44} - 1 & 0 & 0 \\ w_{51} & 0 & 0 & w_{54} & -1 & 0 \\ 0 & 0 & 0 & 0 & w_{65} & w_{66} - 1 \end{pmatrix}.$$

Note that if we cut the vertical edges which are edges e_2 and e_5 , we should specialize all the variables with 2 or 5 in them to be 0. This yields the edge zeta function of the subgraph with the vertical edge removed, and incidentally diagonalizes the matrix W . We call this “**fission**”. The edge zeta is particularly suited to keeping track of such fission.

Next we give a version of **Bass’s proof of the Ihara determinant formula** using the preceding theorem. In what follows, n is the number of vertices of X and m is the number of unoriented edges of X .

First define some matrices. Set $J = \begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix}$. Then define the $n \times 2m$ **start matrix** S and the $n \times 2m$ **terminal matrix** T by setting

$$s_{ve} = \begin{cases} 1, & \text{if } v \text{ is starting vertex of edge } e, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$t_{ve} = \begin{cases} 1, & \text{if } v \text{ is terminal vertex of edge } e, \\ 0, & \text{otherwise.} \end{cases}$$

PROPOSITION 1. (**Some Matrix Identities**) Using the preceding definitions, the following formulas hold. We write tM for the transpose of the matrix M .

1) $SJ = T, \quad TJ = S.$

2) If A is the adjacency matrix of X and $Q + I_n$ is the diagonal matrix whose j th diagonal entry is the degree of the j th vertex of X , then $A = S {}^tT$, and $Q + I_n = S {}^tS = T {}^tT$.

3) The matrix W_1 from Definition 2.2 satisfies $W_1 + J = {}^tTS$.

PROOF. 1) comes from the fact that the starting (terminal) vertex of edge e_j is the terminal (starting) vertex of edge $e_{j+|E|}$, according to our edge numbering system from formula (2.1).

2) Consider

$$(S {}^tT)_{a,b} = \sum_e s_{ae} t_{be}.$$

The right hand side is the number of edges e such that a is the initial vertex and b is the terminal vertex of e , which is the a, b entry of A . Similar arguments prove the second formula.

3) We have

$$({}^tTS)_{ef} = \sum_v t_{ve} s_{vf}.$$

The sum is 1 iff edge e feeds into edge f , even if $f = e^{-1}$. □

Bass's Proof of the Ihara Determinant Formula (2.3).

PROOF. In the following identity all matrices are $(n + 2m) \times (n + 2m)$, where the 1st block is $n \times n$, if n is the number of vertices of X and m is the number of unoriented edges of X . Use the preceding proposition to see that

$$\begin{aligned} & \begin{pmatrix} I_n & 0 \\ {}^tT & I_{2m} \end{pmatrix} \begin{pmatrix} I_n(1 - u^2) & Su \\ 0 & I_{2m} - W_1u \end{pmatrix} \\ &= \begin{pmatrix} I_n - Au + Qu^2 & Su \\ 0 & I_{2m} + Ju \end{pmatrix} \begin{pmatrix} I_n & 0 \\ {}^tT - {}^tSu & I_{2m} \end{pmatrix}. \end{aligned}$$

Take determinants to obtain

$$(1 - u^2)^n \det(I - W_1u) = \det(I_n - Au + Qu^2) \det(I_{2m} + Ju).$$

To finish the proof of formula (2.3), observe that

$$I + Ju = \begin{pmatrix} I & Iu \\ Iu & I \end{pmatrix}$$

implies

$$\begin{pmatrix} I & 0 \\ -Iu & I \end{pmatrix} (I + Ju) = \begin{pmatrix} I & Iu \\ 0 & I(1 - u^2) \end{pmatrix}.$$

Thus $\det(I + Ju) = (1 - u^2)^m$. Since $r - 1 = m - n$, for a connected graph, the Ihara formula (2.3) follows. □

4. Path Zeta Functions

Here we look at a zeta function invented by Stark. It has several advantages over the edge zeta. It can be used to compute the edge zeta with smaller determinants. It gives the edge zeta for a graph in which an edge has been fused; i.e., shrunk to one vertex.

First recall that the fundamental group of X can be identified with the group generated by the edges left out of a spanning tree T of X . We call the oriented versions of these **edges left out of the spanning tree T (or “cut” edges of T)** (and their inverses)

$$e_1, \dots, e_r, e_1^{-1}, \dots, e_r^{-1}.$$

Call the remaining (oriented) **edges (and their inverses) in the spanning tree T**

$$t_1, \dots, t_u, t_1^{-1}, \dots, t_u^{-1}.$$

DEFINITION 4.1. The $2r \times 2r$ **path matrix** Z has ij entry given by the complex variable z_{ij} if $e_i \neq e_j^{-1}$ and by 0 if $e_i = e_j^{-1}$.

Note that the path matrix Z has only one zero entry in each row unlike the edge matrix W from Definition 3.1 which is rather sparse.

Next we imitate the definition of the edge zeta function. So we define the **path norm** for a prime path $C = a_1 \cdots a_s$, where $a_i \in \{e_1^{\pm 1}, \dots, e_s^{\pm 1}\}$ as

$$N_F(C) = z_{a_1 a_2} \cdots z_{a_{s-1} a_s} z_{a_s a_1}.$$

Then the **path zeta** is defined to be

$$\zeta_F(Z, X) = \prod_{[P]} (1 - N_F(P))^{-1}.$$

where the product is over primes $[P]$.

The determinant formula for the path zeta is proved similarly to that for the edge zeta and it says

$$\zeta_F(Z, X)^{-1} = \det(I - Z).$$

Next we want to give a way to get the edge zeta out of the path zeta. To do this requires a procedure called **specializing the path matrix to the edge matrix**. Use the notation above for the edges left out of the spanning tree T and the edges of T . A prime cycle P is first written as a product of generators of the fundamental group and then as a product of actual edges e_i and t_k . Do this by inserting $t_{k_1} \cdots t_{k_s}$ which is the unique non backtracking path on T joining the terminal vertex of e_i and the starting vertex of e_j if e_i and e_j are successive cut edges in P . Now **specialize the path matrix Z to $Z(W)$ with entries**

$$(4.1) \quad z_{ij} = w_{e_i t_{k_1}} w_{t_{k_1} t_{k_2}} \cdots w_{t_{k_{n-1}} t_{k_n}} w_{t_{k_n} e_j}.$$

Then the path zeta function at $Z(W)$ specializes to the edge zeta function

$$\zeta_F(Z(W), X) = \zeta_E(W, X).$$

M. Horton [15] has a Mathematica program to do the specialization in formula (4.1).

Example. The Dumbbell Again.

Recall that the edge zeta of the dumbbell graph of Figure 5 was evaluated by a 6×6 determinant. The path zeta requires a 4×4 determinant. Take the spanning tree to be the vertical edge. That is really the only choice here. One finds using the determinant formula for the path zeta and the specialization of the path to edge zeta:

$$(4.2) \quad \zeta_E(W, X)^{-1} = \det \begin{pmatrix} w_{11} - 1 & w_{12}w_{23} & 0 & w_{12}w_{26} \\ w_{35}w_{51} & w_{33} - 1 & w_{35}w_{54} & 0 \\ 0 & w_{42}w_{23} & w_{44} - 1 & w_{42}w_{26} \\ w_{65}w_{51} & 0 & w_{65}w_{54} & w_{66} - 1 \end{pmatrix}.$$

If we shrink the vertical edge to a point (which we call “fusion”), the edge zeta of the new graph is obtained by replacing any $w_{x2}w_{2y}$ (for $x, y = 1, 3, 4, 6$) which appear in formula (4.2) by w_{xy} and any $w_{x5}w_{5y}$ (for $x, y = 1, 3, 4, 6$) by w_{xy} . This gives the zeta function of the new graph obtained from the dumbbell, by fusing the vertical edge.

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