Zeta Functions of Complexes

Ming-Hsuan Kang

and

Winnie Li

Pennsylvania State University

1. The Ihara vertex zeta function of a graph

- X : connected undirected finite graph
- Count backtrackless and tailless cycles.

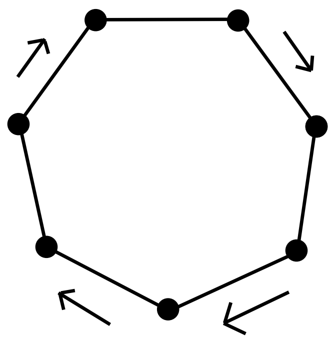

Figure 1: without tail

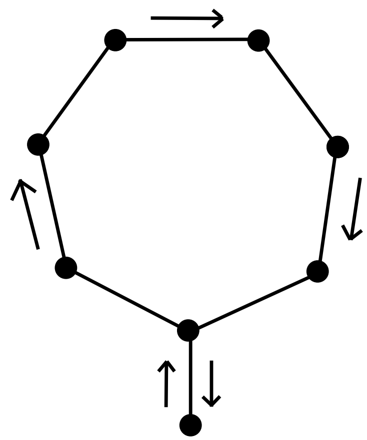

Figure 2: with tail

- *Primitive* cycle: not repeating another cycle more than once.

The Ihara vertex zeta function of X is defined as

$$Z_0(X;u) = \prod_{[C]} \frac{1}{1-u^{l(C)}} \ ,$$

where $[C]$ runs through all equiv. classes of primitive backtrackless and tailless cycles C, and $l(C)$ is the length of C.

Note that

$$u\frac{d}{du}\log Z_0(X;u) = \sum_{n\geq 1} N_n u^n,$$

where N_n is the number of backtrackless and tailless cycles of length n. Therefore

$$Z_0(X;u) = \prod_{[C]} \frac{1}{1-u^{l(C)}} = \exp\left(\sum_{n\geq 1} \frac{N_n}{n} u^n\right).$$

2. Properties of Ihara zeta function for regular graphs

- Ihara (1968): Let X be a finite $(q+1)$-regular graph. Then its zeta function $Z_0(X,u)$ is a rational function of the form

$$Z_0(X;u) = \frac{(1-u^2)^{\chi(X)}}{\det(I - Au + qu^2 I)} ,$$

where $\chi(X) = \#V - \#E$ is the Euler characteristic of X and A is the adjacency matrix of X.

- X is Ramanujan if and only if $Z_0(X,u)$ satisfies RH, i.e. the nontrivial poles of $Z_0(X,u)$ all have absolute value $q^{-1/2}$.

Recall

- The trivial eigenvalues of X are $\pm(q+1)$, of multiplicity one.
- X is called a *Ramanujan graph* if the nontrivial eigenvalues λ satisfy the bound
$$|\lambda| \leq 2\sqrt{q},$$
i.e. the roots of $1 - \lambda u + qu^2$ have absolute value $q^{-1/2}$.
Alon-Boppana: This eigenvalue bound is best possible.

3. The Hashimoto edge zeta function of a graph

Endow two orientations on each edge of a finite graph X. The neighbors of $\overrightarrow{e}$ are the directed edges starting from the ending vertex of $\overrightarrow{e}$ and not equal to the opposite of $\overrightarrow{e}$.

Associate the edge adjacency matrix A_e.

The Hashimoto edge zeta function $Z_1(X, u)$ counts backtrackless and tailless oriented edge cycles, hence the same as $Z_0(X, u)$. Since $N_n = \mathrm{Tr} A_e^n$, we get

$$Z_0(X, u) = Z_1(X, u) = \frac{1}{\det(I - A_e u)}.$$

Another viewpoint of the Ihara's Theorem:

$$(1 - u^2)^{\chi(X)} = \frac{\det(I - Au + qu^2 I)}{\det(I - A_e u)}.$$

When q is a power of a prime p, the $(q+1)$-regular tree

$$\mathcal{T} = \mathrm{PGL}_2(F)/\mathrm{PGL}_2(\mathcal{O}_F),$$

where F is a p-adic field with q elements in its residue field and $\mathcal{O}_F$ is its ring of integers.

vertices : $\mathrm{PGL}_2(\mathcal{O}_F)$-cosets
vertex adjacency operator A : Hecke operator on

$$\mathrm{PGL}_2(\mathcal{O}_F)\begin{pmatrix}1 & 0\\ 0 & \pi\end{pmatrix}\mathrm{PGL}_2(\mathcal{O}_F).$$

directed edges : $\mathcal{I}$-cosets ($\mathcal{I}$ is the Iwahori subgroup of $\mathrm{PGL}_2(\mathcal{O}_F)$)
edge adjacency operator A_e : Iwahori-Hecke operator on $\mathcal{I}\begin{pmatrix}1 & 0\\ 0 & \pi\end{pmatrix}\mathcal{I}$.

$X = X_\Gamma = \Gamma\backslash\mathrm{PGL}_2(F)/\mathrm{PGL}_2(\mathcal{O}_F) = \Gamma\backslash\mathcal{T}$ for a torsion free discrete cocompact subgroup Γ of $PGL_2(F)$.

4. The building on $\mathbf{PGL}_3(F)$

- F, $\mathcal{O}_F$, π : the same as before
- $G = \mathrm{PGL}_3(F)$, $K = \mathrm{PGL}_3(\mathcal{O}_F)$
- The Bruhat-Tits building $\mathcal{B} = G/K$ is a 2-dim'l simplicial complex. The chambers are the 2-simplices, their edges are the 1-simplices, and the vertices are the 0-simplices.
- $\mathcal{B}$ is $(q+1)$-regular, namely, each edge is shared by exactly $q+1$ chambers.
- Topologically $\mathcal{B}$ is simply connected, so it is the universal cover of its finite quotients, called 3-hypergraphs/2-dim'l complexes.

5. Parametrizations of the simplices in $\mathcal{B}$

- $\sigma = \begin{pmatrix} & 1 & \\ & & 1 \\ p & & \end{pmatrix}$. Have a filtration of K:

$$K \supset E := K \cap \sigma K \sigma^{-1} \supset B := K \cap \sigma K \sigma^{-1} \cap \sigma^{-1} K \sigma.$$

- vertices $\leftrightarrow$ K-cosets

  Each vertex gK has a *type* in $\mathbb{Z}/3\mathbb{Z}$ given by $\tau(gK) := \mathrm{ord}_\pi(\det g)$ mod 3.

- The type of an edge $gK \to g'K$ is $\tau(g'K) - \tau(gK) = 1$ or 2.
- type one edges $\leftrightarrow$ E-cosets
- chambers $\leftrightarrow$ B-cosets such that gB, $g\sigma B$ and $g\sigma^2 B$ represent the same chamber.

6. Operators on $\mathcal{B}$

The K-double cosets define Hecke operators acting on $L^2(G/K)$. They are polynomials in

$$A_1 = K \begin{pmatrix} 1 & & \\ & 1 & \\ & & \pi \end{pmatrix} K \quad \text{and} \quad A_2 = K \begin{pmatrix} 1 & & \\ & \pi & \\ & & \pi \end{pmatrix} K.$$

The B-double cosets define Iwahori-Hecke operators acting on $L^2(G/B)$. Denote by L_B the operator supported on $Bt_2\sigma^2 B$, where

$$t_2 = \begin{pmatrix} & & \pi^{-1} \\ & 1 & \\ \pi & & \end{pmatrix}.$$

7. Finite quotients of $\mathcal{B}$

Γ : a torsion free discrete subgroup of G with compact quotient.

$X = X_\Gamma = \Gamma\backslash G/K = \Gamma\backslash\mathcal{B}$

Two assumptions on Γ:

(I) $\mathrm{ord}_\pi \det \Gamma \subset 3\mathbb{Z}$ so that Γ identifies vertices of the same type.

(II) Γ is *regular*, that is, the centralizer in Γ of any nonidentity element $\gamma \in \Gamma$ is a torus.

Division algebras of degree 9 yield many such Γ's.

Goal: Find a closed form expression of the vertex zeta function of X analogous to graph zeta functions.
Previously considered by Deitmar, and Deitmar-Hoffman, partial results.

8. The main results

The type one vertex zeta function on X is defined as

$$Z_{0,1}(X,u) = \prod_{[\mathfrak{C}]} \frac{1}{1-u^{l_A(\mathfrak{C})}},$$

where $[\mathfrak{C}]$ runs through the equiv. classes of primitive tailless type one cycles in X.

Such cycles traveled in reverse direction are type two cycles with algebraic length doubled. Define

$$Z_0(X,u) = Z_{0,1}(X,u)Z_{0,2}(X,u) = Z_{0,1}(X,u)Z_{0,1}(X,u^2).$$

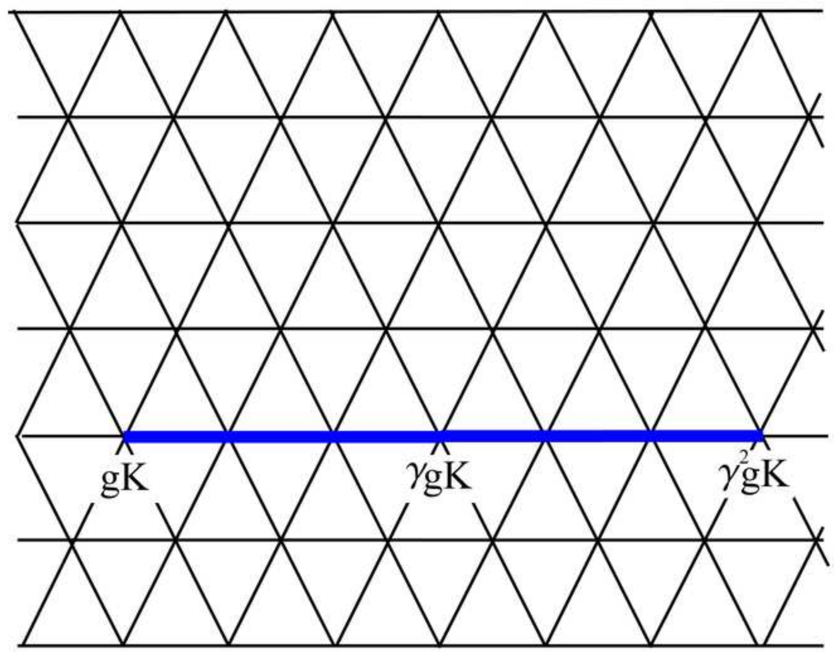

Figure 3: tailless

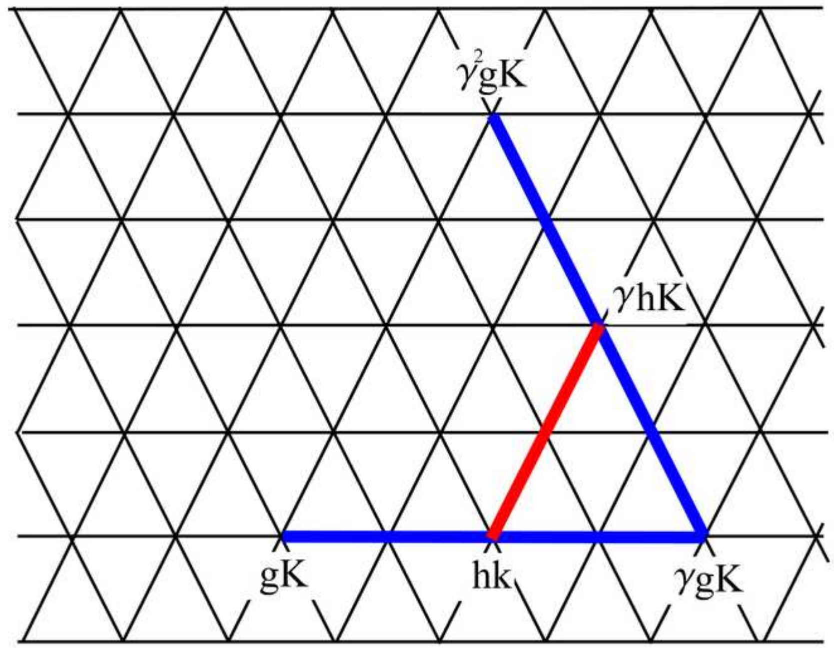

Figure 4: with tail

Main Theorem

(1) $Z_0(X,u)$ *is a rational function given by*

$$Z_0(X,u) = \frac{(1-u^3)^{\chi(X)}}{\det(I - A_1 u + qA_2 u^2 - q^3 u^3 I)\det(I + L_B u)},$$

where $\chi(X) = \#V - \#E + \#C$ *is the Euler characteristic of* X*.*

(2) X *is a Ramanujan complex if and only if* $Z_0(X,u)$ *satisfies the RH.*

Recall that

- The trivial eigenvalues of $I - A_1 u + qA_2 u^2 - q^3 u^3 I$ are 1, q^{-1}, q^{-2} as well as their multiples by the cubic roots of 1.

- X is a 2-dim'l *Ramanujan complex* if the nontrivial eigenvalues of $I - A_1 u + qA_2 u^2 - q^3 u^3 I$ all have absolute value q^{-1}.

  Li (2004): Such bounds for eigenvalues of A_1 and A_2 are best possible.

  In this case the nontrivial roots of $\det(1 + L_B u)$ have absolute value $q^{-1/2}$, proved by Kang-Li-Wang.

9. A sketch of the proof

- Partition the homotopy geodesic cycles on X into sets parametrized by the conjugacy classes $[\gamma]$ of Γ; the set indexed by $[\gamma]$ consists of cycles $\kappa_\gamma(gK)$ with starting point $gK \in C_\Gamma(\gamma)\backslash G/K$ and ending point γgK. Here $C_\Gamma(\gamma)$ is the centralizer of γ in Γ.
- Give an algebraic criterion of tailless cycles.
- Compute the number of type one cycles of given length in each set $[\gamma]$, with and without tails. Need to separate γ into three cases, according to the field generated by its eigenvalues being F, quadratic ramified or unramified extension of F. (Γ does not contain elements whose eigenvalues generate a cubic extension of F.) Very complicated.

- Put together, this allows us to express

$$\frac{(1-u^3)^{\chi(X)}}{\det(I - A_1 u + qA_2 u^2 - q^3 u^3 I)}$$

as $L_{0,1}(X, u)$ times "something extra".

- Show that the zeta function of the type one tailless closed galleries in X is $L_2(X, u) = 1/\det(I - L_B u)$. The boundaries of such galleries are type one tailless cycles in X. Characterize the chambers which will lie in a tailless type one closed gallery. This allows us to compare $L_2(X, -u)$ with $L_{0,2}(X, u)$, leading to the conclusion that "something extra" is $L_{0,2}(X, u)/L_2(X, -u)$.