Math 260A — Mathematical Logic — Scribe Notes
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1 Proof systems for first order logic

In propositional logic, the simplest proof system is truth tables. Unfortunately in first order logic there is no analog; the truth tables would be infinite. Instead, we dive right into Frege-style systems. In first order logic, these are also called Hilbert-style systems.

1.1 Frege proofs

Axioms We use the following axioms:

- All axioms from propositional Frege are allowed, such as $\varphi \to (\psi \to \varphi)$. Note that φ, for example, is now allowed to be any formula in first order logic.
- For any formula φ, and any term t, we get axioms $\varphi(t) \to \exists x \varphi(x)$ and $\forall x \varphi(x) \to \varphi(t)$. There are concerns that t should be substitutable for t, but we will mostly ignore these.
- If $=$ is in the language, then the *equality axioms* are available:
  - $\forall x(x = x)$
  - $\forall x \forall y(x = y \to y = x)$
  - $\forall x \forall y \forall z(x = y \land y = z \to x = z)$
  - $\forall x_1 \ldots \forall x_k \forall y_1 \ldots \forall y_k(f(x_1, \ldots, x_k) = f(y_1, \ldots, y_k))$ for all functions f of arity k.
  - $\forall x_1 \ldots \forall x_k \forall y_1 \ldots \forall y_k(P(x_1, \ldots, x_k) \to P(y_1, \ldots, y_k))$ for all relations P of arity k.

Rules of inference

- Modus Ponens: $\dfrac{\varphi \qquad \varphi \to \psi}{\psi}$

- We also need two inference rules for quantifiers. If x is not free in ψ, then we have:
$\dfrac{\psi \to \varphi}{\psi \to \forall x\varphi} \qquad \dfrac{\varphi \to \psi}{\exists x\varphi \to \psi}$

1.2 Soundness

Definition Say that $\Gamma \vdash \varphi$ if there is a sequence of formulas, coming from Γ, the axioms, and inferences, ending at φ.

Theorem (Soundness) If $\Gamma \vdash \varphi$, then $\Gamma \vDash \varphi$.

Proof Assume we have a proof P of φ from Γ. Let $\mathcal{M}$ be a structure in which Γ is valid. Recall this means that, for all object assignments σ, for all $\gamma \in \Gamma$, $\mathcal{M} \vDash \gamma[\sigma]$.

We prove by induction (on the number of lines in P) that, if $\psi \in P$, then for all σ, $\mathcal{M} \vDash \psi[\sigma]$.

The base cases (axioms) are easy, as is modus ponens.

We will look at the inference rule about the existential quantifier to see the idea. The soundness of the other rule is similar.

To prove that

$$\frac{\varphi \to \psi}{\exists x\varphi \to \psi}$$

is sound, we argue by contrapositive. Suppose we have a structure $\mathcal{M}$ and an object assignment σ so that $\mathcal{M} \nvDash (\exists x\varphi \to \psi)[\sigma]$. By the definition, this means that $\mathcal{M} \vDash \exists x\varphi[\sigma]$ and $\mathcal{M} \nvDash \psi[\sigma]$.

The former means there is some $m \in \mathcal{M}$ so that $\mathcal{M} \vDash \psi[\sigma(m/x)]$.

Define $\sigma' = \sigma(m/x)$. Since x is not free in ψ, $\mathcal{M} \nvDash \psi[\sigma']$ — the value of x has no bearing on the truth of ψ. For this object assignment, we now see that φ holds, but ψ does not. Thus $\mathcal{M} \nvDash (\varphi \to \psi)[\sigma]$, as desired.

Notice that we needed to change the object assignment σ in order to make this work.

2 Sequent calculus for first order logic

In order to formulate sequent calculus for first order logic, we need to set some conventions for variable names.

We will use $a, b, c, \ldots$ to refer to free variables, and $x, y, z, \ldots$ to refer to bound variables. These are strict conventions, and they have some troubling consequences. Subformulas of formulas are not necessarily formulas themselves. For example, $\forall x(x \leq a)$ is a formula, but $x \leq a$ is not, since x is not bound. Sometimes this is called a *semi-formula*. Similarly, $x + a$ is a *semi-term*, not a proper term.

Note: if φ is a formula, a is a formula, and t is a term, then t is substitutable for a in φ.

2.1 LK

The following axioms and rules of inference define the proof system LK, from the German for Logical Calculus.

Initial sequents The only initial sequent is $A \longrightarrow A$, for any formula A.

Rules of inference

- All inference rules from propositional sequent calculus are included in this system.
- $\forall$-right:
$$\frac{\Gamma \longrightarrow \Delta, A(b)}{\Gamma \longrightarrow \Delta, \forall x A(x)}$$
The variable b, called an eigenvariable, is not allowed to appear anywhere in the lower sequent. In particular, the substitution x/b "captures" all appearances of b in the formula A.
- $\exists$-right:
$$\frac{\Gamma \longrightarrow \Delta, A(t)}{\Gamma \longrightarrow \Delta, \exists x A(x)}$$
Here t is any term, and *may* occur in the lower sequent.
- $\exists$-left:
$$\frac{A(b), \Gamma \longrightarrow \Delta}{\exists x A(x), \Gamma \longrightarrow \Delta}$$
This has the same conditions as $\forall$-right.
- $\forall$-left:
$$\frac{A(t), \Gamma \longrightarrow \Delta}{\forall x A(x), \Gamma \longrightarrow \Delta}$$
Here t is any term, and *may* occur in the lower sequent.

2.2 LK$_{\mathrm{e}}$

The system we just saw is not sufficient for a language including equals, in that it has no rules to handle equality. To overcome this, we introduce LK$_{\mathrm{e}}$, with 5 additional initial sequents.

Additional initial sequents

- $\longrightarrow s = s$
- $s = t \longrightarrow t = s$
- $s = t, t = u \longrightarrow s = u$
- $s_1 = t_1, \ldots, s_k = t_k \longrightarrow f(s_1, \ldots, s_k) = f(t_1, \ldots, t_k)$ for any formula f of arity k.
- $s_1 = t_1, \ldots, s_k = t_k \longrightarrow P(s_1, \ldots, s_k) \to P(t_1, \ldots, t_k)$ for any relation P of arity k.

These allow us to work with sequents about equality.

Example To prove $\forall x \forall y (x = y \to y = x)$, we may do this:

$$\dfrac{\dfrac{\dfrac{a = b \longrightarrow b = a}{\longrightarrow a = b \to b = a}\ \to\text{-right}}{\longrightarrow \forall y (a = y \to y = a)}\ \forall\text{-right}}{\longrightarrow \forall x \forall y (x = y \to y = x)}\ \forall\text{-right}$$