Exponential functions

Define exponential function. An exponential function is a function of the form
\[ f(x) = b^x, \] where \( b > 0 \) and \( b \neq 1 \). The number \( b \) is called the base.

Examples include \( f(x) = e^x \) and \( g(x) = 0.5^x \). Notice the variable appears in the exponent. With each increase of one in the variable the value of the function, \( y \), increases by a factor of \( b \), i.e., \( b^{x+1} = b(b^x) \). For the inputs 0, 1, 2, 3, \ldots \( f(x) = 1, e, e^2, e^3, \ldots \) and \( g(x) = 1, 0.5, 0.25, 0.125, \ldots \)

If \( b > 1 \), then our sequence grows; if \( 0 < b < 1 \), then our sequence decays.

We exclude base values of zero, one, and negative numbers. Why? Our definition of an exponential function does not limit the variable to integer values. Rational exponents are well defined, e.g., \( 4^{3/2} = \sqrt{4^3} \) or \((\sqrt{4})^3\), so \( 4^{3/2} = 8 \). In fact, our definition includes irrational input values, e.g., \( 3^{\sqrt{2}} \). Since we can estimate \( \sqrt{2} \) with a rational number, we can estimate \( 3^{\sqrt{2}} \) as closely as we desire.

Graphs of exponential functions, domain, range, intercepts, asymptotes. The domain of an exponential function \( y = b^x \) is the set of all real numbers. The range of an exponential function is the set of positive real numbers.

Since \( b^0 = 1 \), the \( y \)-intercept is (0,1). There are no \( x \)-intercepts (zeros).

The \( x \)-axis is a horizontal asymptote. The graph of \( y = b^x \) is increasing on its entire domain (and one-to-one) if \( b > 1 \) and decreasing if \( 0 < b < 1 \). Therefore, the inverse function exists. Sketch graphs of exponential functions \( f(x) = 2^x \) and \( g(x) = 0.5^x \). What about the graph of \( h(x) = 2^{-x} \)?

Translations, reflections and scalings. We can translate the graph \( h \) units horizontally, \( k \) units vertically, and scale by a factor of \( a \) vertically, so that the graph \( y = ab^{x-h} + k \) passes through the point \((h, a + k)\). The horizontal asymptote becomes \( y = k \). Sketch the graph of \( y = 5 - 2e^{x-1} \). This is a translation of \( y = e^x \) by 1 unit right, reflection across the \( x \)-axis, vertical scaling by a factor of 2, and translation up by 5 units. The horizontal asymptote is \( y = 5 \) and the graph passes through the point \((1,3)\).

Natural exponential function. One particular base for exponential functions is most useful. This base is the irrational number \( e \), where \( e = 2.718\ldots \). The function \( y = e^x \) is called the natural exponential function. The graph of \( y = e^x \) lies between the graphs of \( y = 2^x \) and \( y = 3^x \).

Hyperbolic functions. Hyperbolic functions are certain special combinations of \( e^x \) and \( e^{-x} \) that play a role in engineering and physics. The hyperbolic sine is defined as \( \sinh(x) = \frac{e^x - e^{-x}}{2} \), and hyperbolic cosine is defined as...
\[ \cosh(x) = \frac{e^x + e^{-x}}{2}. \]

There are similarities between the hyperbolic sine and cosine functions and the trigonometric sine and cosine functions. For example, the hyperbolic sine function is \textit{odd} just like the trigonometric sine function, and the hyperbolic cosine function is \textit{even} just like the trigonometric cosine function. Show that \( \cosh(-x) = \cosh(x) \).

The hyperbolic sine and cosine functions satisfy some of the same identities satisfied by the trigonometric sine and cosine functions, e.g.,

\[ \tanh(x) = \frac{\sinh(x)}{\cosh(x)}. \]

Prove this identity. The Pythagorean identity for the hyperbolic sine and cosine is slightly different than its trigonometric counterpart, \( \cosh^2(x) - \sinh^2(x) = 1 \). Prove this identity.