Definition. A sequence $\langle x_n \rangle$ is **Cauchy** if for each positive number ε there is a natural number N such that if $m, n > N$, then

$$|x_m - x_n| < \varepsilon.$$

Example: The sequence $\langle n \rangle$ is neither Cauchy nor bounded.

Facts: A Cauchy sequence is bounded. Even better, a Cauchy sequence converges.

Theorem. If $\langle x_n \rangle$ and $\langle y_n \rangle$ are Cauchy sequences, then

1. $\langle x_n + y_n \rangle$ is a Cauchy sequence,
2. $\langle x_n y_n \rangle$ is a Cauchy sequence.

Proof. (1) Let $\langle x_n \rangle$ and $\langle y_n \rangle$ be Cauchy sequences. For any $\varepsilon > 0$ there exist $N_1, N_2 \in \mathbb{N}$ such that

(i) If $m, n > N_1$, then $|x_m - x_n| < \frac{\varepsilon}{2}$.

(ii) If $m, n > N_2$, then $|y_m - y_n| < \frac{\varepsilon}{2}$.

Let $N = \max\{N_1, N_2\}$. If $m, n > N$, then by the triangle inequality,

$$\begin{aligned}|(x_m + y_m) - (x_n + y_n)| &= |(x_m - x_n) + (y_m - y_n)| \\ &\leq |x_m - x_n| + |y_m - y_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

By definition, this proves that $\langle x_n + y_n \rangle$ is a Cauchy sequence.

(2) Let $\langle x_n \rangle$ and $\langle y_n \rangle$ be Cauchy sequences. Then there exists a number B such that

$$|x_n| \leq B \text{ and } |y_n| \leq B$$

for all n.For any $\varepsilon > 0$ there exist $N_1, N_2 \in \mathbb{N}$ such that

(i) If $m, n > N_1$, then $|x_m - x_n| < \frac{\varepsilon}{2B}$.

(ii) If $m, n > N_2$, then $|y_m - y_n| < \frac{\varepsilon}{2B}$.

Let $N = \max\{N_1, N_2\}$. If $m, n > N$, then by the triangle inequality,

$$\begin{aligned} |x_m y_m - x_n y_n| &= |x_m y_m - x_n y_m + x_n y_m - x_n y_n| \\ &= |(x_m - x_n) y_m + x_n (y_m - y_n)| \\ &\leq |x_m - x_n| |y_m| + |x_n| |y_m - y_n| \\ &< \frac{\varepsilon}{2B} B + B \frac{\varepsilon}{2B} = \varepsilon. \end{aligned}$$

By definition, this proves that $\langle x_n y_n \rangle$ is a Cauchy sequence.

Definition. Let $S \subseteq \mathbb{R}$ and let $x \in \mathbb{R}$. Then x is an **accumulation point** of S if for every $\varepsilon > 0$ the set $S \cap (x - \varepsilon, x + \varepsilon)$ is infinite.

Example. The set $\mathbb{Z} \subseteq \mathbb{R}$ has no accumulation points.