NON-UNIQUENESS OF SOLUTIONS TO THE CONFORMAL FORMULATION

MICHAEL HOLST AND CALEB MEIER

ABSTRACT. It is well known that solutions to the conformal formulation of the Einstein constraint equations are unique in the event that the mean curvature is constant (CMC) or near constant (near-CMC). However, the new far-from-constant mean curvature (far-from-CMC) solution constructions due to Holst, Nagy, and Tsogtgerel and to Maxwell in 2009, and to Gicquad and collaborators in 2010, are based on degree theory rather than the contraction arguments used originally by Isenberg and Moncrief in 1996 for the near-CMC case, and hence little is known about the uniqueness of far-from-CMC constructions. In fact, Maxwell recently demonstrated that solutions are non-unique in the far-from-CMC case for certain families of low regularity mean curvatures. In this article, we investigate uniqueness properties of solutions to the Einstein constraint equations on a closed manifold using standard tools in bifurcation theory. For positive, constant scalar curvature and constant mean curvature, we first demonstrate existence of a critical energy density for the Hamiltonian constraint with unscaled matter sources. We then show that for this choice of energy density, the linearization of the elliptic system develops a one-dimensional kernel in both the CMC and non-CMC (near and far) cases. Using a Liapunov-Schmidt reduction and some standard techniques from nonlinear analysis, we demonstrate that solutions to the conformal formulation with unscaled data are non-unique by determining an explicit solution curve and analyzing its behavior in the neighborhood of a particular solution.

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1. Introduction

In this paper we demonstrate that solutions to the Einstein constraint equations on a 3-dimensional closed manifold $(\mathcal{M}, \hat{g}_{ab})$ with no conformal killing field are non-unique. More specifically, we show that solutions to the conformal formulation of the constraint equations with an unscaled matter source on $(\mathcal{M}, \hat{g})$ exhibit non-uniqueness in the case that the scalar curvature is positive and constant. Letting $\hat{k}_{ab}$ be a $(0,2)$ tensor and $\hat{R}$ and $\hat{D}$ be the scalar curvature and connection associated with $\hat{g}_{ab}$, the constraint equations take the form

$$\hat{R} + \hat{k}^2 - \hat{k}^{ab}\hat{k}_{ab} = 2\kappa\hat{\rho}, \tag{1.1}$$

$$\hat{D}^a\hat{k} + \hat{D}_b\hat{k}^{ab} + \kappa\hat{j}^a = 0. \tag{1.2}$$

Equation (1.1) is known as the **Hamiltonian Constraint** and (1.2) is known as the **momentum constraint**.

Equations (1.1) and (1.2) form a system of coupled elliptic partial differential equations. When one attempts to solve the constraint equations they are faced with the problem of having twelve pieces of initial data and only four constraints. One solution to this problem is to attempt to parametrize solutions to (1.1) and (1.2) by formulating the constraints so that eight pieces of initial data are freely specifiable while four are determined by (1.1)-(1.2). The conformal transverse traceless (CTT) decomposition and the conformal thin sandwich method (CTS method) are standard ways of doing this. The extended conformal thin sandwich method (XCTS method) is popular among numerical relativists and reformulates (1.1) and (1.2) as a coupled system of 5 elliptic equations. In the CTT method one decomposes $\hat{k}_{ab}$ into its trace or mean curvature and trace free part and then scales this trace free tensor, the metric $\hat{g}_{ab}$ and the source terms $\hat{\rho}$ and $\hat{j}$ by judicious choices of some power of a positive, smooth function ϕ. The choice of scaling power for each term is typically made to simplify the analysis of the resulting system. In particular, one chooses powers to eliminate terms involving $(D_a\phi)/\phi$ and so that the system decouples when the mean curvature is constant.

It is well known that solutions to the CTT formulation of the constraint equations with scaled data sources are unique in the event that the mean curvature is constant or near constant [9, 10, 1, 7, 8]. However, given that far-from-constant mean curvature solutions are constructed using a variation of the Schauder fixed-point technique as opposed to the contraction mapping theorem (cf. [8]), little is known about the uniqueness of far-from-CMC solutions. In fact, in [12] Maxwell demonstrated that solutions of the CTT formulation of the constraint equations are non-unique in the far-from-CMC case for certain families of low regularity mean curvatures. However, it should be noted that the discontinuous mean curvature functions considered by Maxwell in [12] fall outside of the current far-from-CMC solution framework presented by Holst et al. in [8].

In [14] Pfeiffer and York provided numerical evidence for non-uniqueness of the XCTS method on an asymptotically Euclidean manifold. In [4] O'Murchadha et al. conjectured that the non-uniqueness demonstrated by Pfeiffer and York was related to the fact that certain terms in the momentum constraint related to the lapse function have the "wrong sign", which prevents an application of the maximum principle. To support their claim, the authors of [4] analyzed a simplified system corresponding to a spherically symmetric constant density star and explicitly constructed two branches of solutions. In their analysis they proved that solutions to the Hamiltonian constraint (1.1) with an unscaled matter source are non-unique. Then in [15], Walsh generalized the work of O'Murchadha et al. by applying a Liapunov-Schmidt reduction to both the Hamiltonian constraint with an unscaled matter source and to the XCTS system on an asymptotically Euclidean manifold. However, Walsh relied on the assumption of the existence of a critical density for which the linearization of these two systems developed a one-dimensional kernel. Here we extend the work of Walsh by applying a Liapunov Schmidt reduction to the CTT formulation of the constraint equations on a closed manifold. We explicitly construct a critical, constant density in the event that the scalar curvature is positive and constant and the transverse traceless tensor has constant magnitude. For this particular density, we then show that solutions to the CTT formulation with an unscaled density are non-unique.

As in [4, 15], we consider a less standard conformal formulation of the constraints by allowing unscaled matter sources ρ and $\mathbf{j}$. However, as opposed to considering the CTS and XCTS formulations as in [14, 4, 15], we consider the CTT formulation. By decomposing our initial data

$$\hat{k}_{ab} = \hat{l}_{ab} + \frac{1}{3}\hat{g}_{ab}\hat{\tau}, \tag{1.3}$$

where $\hat{\tau} = \hat{k}_{ab}\hat{g}^{ab}$ is the trace and $\hat{l}_{ab}$ is the traceless part, making the following conformal rescaling

$$\hat{g}_{ab} = \phi^4 g_{ab}, \quad \hat{l}_{ab} = \phi^{-10} l^{ab}, \quad \hat{\tau} = \tau, \tag{1.4}$$

and then decomposing

$$l_{ab} = (\sigma_{ab} + (\mathcal{L}\mathbf{w})_{ab}), \tag{1.5}$$

where $D_a\sigma^{ab} = 0$ and

$$(\mathcal{L}\mathbf{w})^{ab} = D^a w^b + D^b w^a - \frac{2}{3}(D_c w^c)g^{ab}$$

is the **conformal Killing operator**, we obtain the following unscaled conformal reformulation of (1.1) and (1.2) that we will analyze

$$\begin{aligned} -\Delta\phi + \frac{1}{8}R\phi + \frac{1}{12}\tau^2\phi^5 - \frac{1}{8}(\sigma_{ab} + (\mathcal{L}\mathbf{w})_{ab})(\sigma^{ab} + (\mathcal{L}\mathbf{w})^{ab})\phi^{-7} - 2\pi\rho\phi^5 = 0, \\ - D_b(\mathcal{L}\mathbf{w})^{ab} + \frac{2}{3}D^a\tau\phi^6 + \kappa j^a\phi^{10} = 0. \end{aligned} \tag{1.6}$$

Our non-uniqueness results for (1.6) are of interest for a number or reasons. Most immediately, our analysis shows that the formulation (1.6) is unfavorable due to the non-uniqueness of solutions. Therefore, for a given system, if the CTT formulation with a scaled matter source leads to a set of constraints that is suitable for analysis, which it usually does, then one should use the scaled formulation. However, it is not always the case that the conformal formulation with scaled matter sources is the ideal formulation for a given source. In the case of the Einstein-scalar field system, the conformal formulation that is most amenable to analysis takes on a form very similar to the system (1.6) [5]. In addition, it is the hope of the authors that these results will provide additional insight into the non-uniqueness phenomena associated with the CTT formulation in the far-from-CMC case [12] and with the non-uniqueness phenomena analyzed by Pfeiffer, York, Walsh and O'Murchadha et al. In particular, the analysis conducted in this article clearly demonstrates the effect that the terms with the "wrong sign" discussed by O'Murchadha et al. have on the non-uniqueness of the conformal formulations of the constraints. In the case of (1.6), the negative sign in front of the term $2\pi\rho\phi^5$ is undesirable given that it prevents the semilinear portion of the Hamiltonian constraint from being monotone and the corresponding energy from being convex. By a maximum principle argument, we will see in section 4 that it is this term that directly contributes to the non-uniqueness properties of (1.6).

The rest of this paper is organized as follows. In section 2 we introduce the function spaces that we will use and some basic concepts from functional analysis. Then we discuss the Liapunov-Schmidt reduction that we use to prove non-uniqueness. The statements of the main results of this paper can be found in section 3. The remainder of this paper is then devoted to proving these results. The foundation for our argument is developed in sections 4 and 5. In section 4 we demonstrate the existence of a critical, constant density ρ_c such that if g_{ab} has positive, constant scalar curvature, $|\sigma|$ is constant and $\mathbf{j}^a = 0$, the Hamiltonian constraint in (1.6) will have a positive solution if $\rho \leq \rho_c$ and will have no positive solution if $\rho > \rho_c$. Then in section 5 we use the properties of ρ_c to show that there exists a function ϕ_c at which the linearizations of the uncoupled Hamiltonian operator (CMC case) and coupled system (non-CMC case) have one-dimensional kernels. The existence of a one-dimensional kernel then allows us to apply the Liapunov-Schmidt reduction in section 7 in the CMC case and in section 8 in the non-CMC case. In particular, in section 7 we determine an explicit solution curve for (1.6) that goes through the point $(\phi_c, 0)$ in the CMC case. An analysis of this curve then implies the non-uniqueness of solutions to (1.6) when the mean curvature is constant. Similarly, in section 8 we also determine an explicit solution curve for the full, uncoupled system (1.6) through a point of the form $((\phi_c, \mathbf{0}), 0)$. Again, an analysis of this curve reveals non-uniqueness in the event that the mean curvature is non-constant.

2. Preliminary Material

In this section we give a brief definition of the function spaces, norms and notation that we will use in this article and then discuss some basic concepts from functional analysis and bifurcation theory that will be necessary going forward.

2.1. Banach Spaces, Hilbert Spaces and Direct Sums.

We introduce the fundamental properties of the function spaces with which we will be working. We will primarily be working with Banach spaces, however at times we will need to consider these spaces as subspaces of a Hilbert space. For convenience, we present the basic definitions of these general spaces and define the direct sum of two vector spaces, which will be necessary in our non-uniqueness analysis.

The basic space that we will be working with is a Banach space, where a **Banach space** X is a complete, normed vector space. If the norm $\|\cdot\|$ on X is induced by an inner product, we say that X is a **Hilbert Space**. One can form new Banach spaces and Hilbert spaces from preexisting spaces by considering the direct sum.

Definition 2.1. *Suppose that X_1 and X_2 are Banach spaces with norms $\|\cdot\|_{X_1}$ and $\|\cdot\|_{X_2}$. Then the direct sum $X_1 \oplus X_2$ is the vector space of ordered pairs (x, y) where $x \in X_1$, $y \in X_2$ and addition and scalar multiplication are carried out componentwise.*

We have the following proposition:

Proposition 2.2. *The vector space $X_1 \oplus X_2$ is a Banach space when given the norm*

$$\|(x, y)\|_{X_1 \oplus X_2} = \left(\|x\|^2_{X_1} + \|y\|^2_{X_2}\right)^{\frac{1}{2}}. \tag{2.1}$$

Proof. This follows from the fact that $\|\cdot\|_{X_1}$ and $\|\cdot\|_{X_2}$ are norms and the spaces X_1 and X_2 are complete with respect to these norms. □

We have a similar proposition for Hilbert spaces.

Proposition 2.3. *Suppose that $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces with inner products $\langle\cdot,\cdot\rangle_{\mathcal{H}_1}$ and $\langle\cdot,\cdot\rangle_{\mathcal{H}_2}$. Then the direct sum $H_1 \oplus H_2$ is a Hilbert space with inner product*

$$\langle (w,x),(y,z)\rangle_{\mathcal{H}_1\oplus\mathcal{H}_2} = \langle w,y\rangle_{\mathcal{H}_1} + \langle x,z\rangle_{\mathcal{H}_2}. \tag{2.2}$$

Proof. That $\langle\cdot,\cdot\rangle_{\mathcal{H}_1\oplus\mathcal{H}_2}$ is an inner product follows from the fact that $\langle\cdot,\cdot\rangle_{\mathcal{H}_1}$ and $\langle\cdot,\cdot\rangle_{\mathcal{H}_2}$ are inner products. The expression

$$\|(u,v),(u,v)\|_{\mathcal{H}_1\oplus\mathcal{H}_2} = \sqrt{\langle (u,v),(u,v)\rangle_{\mathcal{H}_1\oplus\mathcal{H}_2}},$$

is a norm on $\mathcal{H}_1 \oplus \mathcal{H}_2$ that coincides with the norm in Proposition 2.2 in the event that the norms on X_1 and X_2 are induced by inner products. □

See [16] for a more complete discussion about the direct sums of Banach spaces.

2.2. **Function Spaces.** Let E denote a given vector bundle over $\mathcal{M}$. In this paper we will consider the Sobolev spaces $W^{k,p}(E)$, the space of k-differentiable sections $C^k(E)$, and the Hölder spaces $C^{k,\alpha}(E)$ where $k \in \mathbb{N},\ p \geq 1,\ \alpha \in (0,1)$ and E will either be the vector bundle $\mathcal{M}\times\mathbb{R}$ of scalar-valued functions or $\mathcal{T}^r_s\mathcal{M}$, the space of (r,s) tensors. Note that all of these spaces with the following norm definitions are Banach spaces and the space $W^{k,2}(E)$ is a Hilbert space for $k \in \mathbb{N}$.

Fix a smooth background metric g_{ab} and let $v^{a_1,\cdots,a_r}_{b_1,\cdots,b_s}$ be a tensor of type $r+s$. Then at a given point $x \in \mathcal{M}$, we define its magnitude to be

$$|v| = (v^{a_1,\cdots,b_s} v_{a_1,\cdots,b_s})^{\frac{1}{2}}, \tag{2.3}$$

where the indices of v are raised and lowered with respect to g_{ab}. We then define the Banach space of k-differentiable functions $C^k(\mathcal{M}\times\mathbb{R})$ with norm $\|\cdot\|_k$ to be those functions u satisfying

$$\|u\|_k = \sum_{j=0}^{k} \sup_{x\in\mathcal{M}} |D^j u| < \infty,$$

where D is the covariant derivative associated with g_{ab}. Similarly, we define the space $C^k(\mathcal{T}^r_s\mathcal{M})$ of k-times differentiable (r,s) tensor fields to be those tensors v satisfying $\|v\|_k < \infty$.

Given two points $x,y \in \mathcal{M}$, we define $d(x,y)$ to be the geodesic distance between them. Let $\alpha \in (0,1)$. Then we may define the $C^{0,\alpha}$ Hölder seminorm for a scalar-valued function u to be

$$[u]_{0,\alpha} = \sup_{x\neq y} \frac{|u(x)-u(y)|}{(d(x,y))^\alpha}.$$

Using parallel transport, this definition can be extended to (r,s)-tensors v to obtain the $C^{k,\alpha}$ seminorm $[u]_{k,\alpha}$ [2]. This leads us to the following definition of the $C^{k,\alpha}(\mathcal{M}\times\mathbb{R})$ Hölder norm

$$\|u\|_{k,\alpha} = \|u\|_k + [u]_{k,\alpha}$$

for scalar-valued functions, and we may define the $C^{k,\alpha}(\mathcal{T}^r_s\mathcal{M})$ Hölder norm for (r,s) tensors in a similar fashion.

Finally, we will also make use of the Sobolev spaces $W^{k,p}(\mathcal{M}\times\mathbb{R})$ and $W^{k,p}(\mathcal{T}^r_s\mathcal{M})$ where we assume $k \in \mathbb{N}$ and $p \geq 1$. If dV_g denotes the volume form associated with g_{ab}, then the L^p norm of an (r,s) tensor is defined to be

$$\|v\|_p = \left(\int_{\mathcal{M}} |v|^p dV_g\right)^{\frac{1}{p}}. \tag{2.4}$$

We can then define the Banach space $W^{k,p}(\mathcal{M} \times \mathbb{R})$ (resp. $W^{k,p}(\mathcal{T}^r_s\mathcal{M})$) to be those functions (resp. (r,s) tensors) v satisfying

$$\|v\|_{k,p} = \left(\sum_{j=0}^{k} \|D^j v\|_p^p\right)^{\frac{1}{p}} < \infty.$$

The above norms are independent of the background metric chosen. Indeed, given any two metrics g_{ab} and $\hat{g}_{ab}$, one can show that the norms induced by the two metrics are equivalent. For example, if D and $\hat{D}$ are the derivatives induced by g_{ab} and $\hat{g}_{ab}$ respectively, then there exist constants C_1 and C_2 such that

$$C_1\|u\|_{k,\hat{g}} \leq \|u\|_{k,g} \leq C_2\|u\|_{k,\hat{g}},$$

where $\|\cdot\|_{k,g}$ denotes the $C^k(\mathcal{M})$ norm with respect to g. This holds for the $W^{k,p}$ and $C^{k,\alpha}$ norms as well. We also note that the above norms are related through the Sobolev embedding theorem. In particular, the spaces $C^{k,\alpha}$ and $W^{l,p}$ are related in the sense that if n is the dimension of $\mathcal{M}$ and $u \in W^{l,p}$ and

$$k + \alpha < l - \frac{n}{p},$$

then $u \in C^{k,\alpha}$. See [2, 3, 6, 13] for a complete discussion of the Sobolev embedding Theorem, Banach spaces on manifolds, and the above norms.

2.3. **Adjoints and Projection Operators.** Solutions to the coupled system (1.6) satisfy

$$F(x, \mathbf{w}) = 0, \tag{2.5}$$

where $F : X \times Y \to Z$ is a nonlinear operator between Banach spaces. This allows us to use basic tools from functional analysis to analyze our problem. In particular, we will repeatedly need to consider the linearization $D_xF(x, \mathbf{w})$, its adjoint, and projections onto subspaces determined by these operators. Later on in the section when we introduce the Liapunov-Schmidt reduction, we will use the kernel of the linearization $D_xF(x_0, \mathbf{w}_0)$ at a point $(x_0, \mathbf{w}_0)$, the kernel of the adjoint, and projection operators onto these subspaces, to decompose X and Y in a manner that will greatly simplify our analysis. Here we briefly discuss the adjoint and projection operators. See [16] for a more complete discussion of these topics and see Appendix 10.1 for a discussion of Fréchet derivatives.

2.3.1. *The Adjoint and Properties.* Suppose that $\mathcal{H}$ is a Hilbert space with inner product $\langle\cdot,\cdot\rangle$. Then if $A : \mathcal{H} \to \mathcal{H}$ is a linear operator, the Riesz Representation Theorem implies that there exists a unique operator A^* that satisfies

$$\langle Ax, y\rangle = \langle x, A^*y\rangle \quad \text{for all } x, y \in \mathcal{H}. \tag{2.6}$$

If $R(A)$ denotes the range of A and $\ker(A)$ denotes the kernel, then the operator A^* satisfies the following properties:

$$\mathbf{1)} \quad \ker(A^*) = R(A)^\perp \tag{2.7}$$

$$\mathbf{2)} \quad (\ker(A^*))^\perp = \overline{R(A)}. \tag{2.8}$$

2.3.2. *Projection Operators and Fredholm Operators.* Now assume that $X \subset \mathcal{H}$ is a Banach space contained in a Hilbert space $\mathcal{H}$. Given a subspace $V \subset X$, the projection P onto V is a bounded linear operator $P : X \to V$ that satisfies $P^2 = P$. In particular, if V is a finite-dimensional subspace spanned by the orthonormal basis $\hat{v}_1, \cdots, \hat{v}_n$, then we can easily construct the projection onto V by the formula

$$Pu : \sum_{i=1}^{n} \langle u, \hat{v}_i \rangle \hat{v}_i, \tag{2.9}$$

where $u \in X$ and $\langle \cdot, \cdot \rangle$ is the inner product on $\mathcal{H}$. Note that P is just the normal projection operator from $\mathcal{H}$ to V restricted to X.

We end the section by introducing one more definition that will be important in the following section. A **Fredholm operator** is a bounded linear operator $A : X \to Y$ where X and Y are Banach spaces such that $\dim \ker(A)$ and $\dim \ker(A^*)$ are finite-dimensional and $R(A)$ is closed. Given a nonlinear operator $F : U \to Y$ where $U \subset X$, we say that F is a **nonlinear Fredholm operator** if it is Fréchet differentiable on U and $D_x F(x)$ is a Fredholm operator.

Notice that if A is a Fredholm operator, then $\ker(A^*)^\perp = R(A)$ and furthermore, the fact that $\ker(A)$ and $\ker(A^*)$ are finite dimensional allows one to define projection operators P and Q onto $\ker(A)$ and $\ker(A^*)$ to decompose X and Y. As we will see, these properties make Fredholm operators ideal candidates for bifurcation analysis.

2.4. **Elements of Bifurcation Theory.** We now present some basic concepts from bifurcation theory that will be essential in obtaining our non-uniqueness results. In particular, we give a formal definition of a bifurcation point and then present the Liapunov-Schmidt reduction. This reduction allows one to reduce a nonlinear problem between infinite-dimensional Banach spaces to a finite-dimensional or even scalar-valued problem. Therefore it greatly simplifies the analysis and will serve as a basic tool for us going forward. The following treatment is taken from [11] and [**?**].

Suppose that $F : U \times V \to Z$ is a mapping with open sets $U \subset X, V \subset \Lambda$, where X and Z are Banach spaces and $\Lambda = \mathbb{R}$. We let $x \in X$ and $\lambda \in \Lambda$. Additionally assume that $F(x, \lambda)$ is Fréchet differentiable with respect to x and λ on $U \times V$. We are interested in solutions to the nonlinear problem

$$F(x, \lambda) = 0. \tag{2.10}$$

A solution of (2.10) is a point $(x, \lambda) \in X \times \Lambda$ such that (2.10) is satisfied.

Definition 2.4. *Suppose that (x_0, λ_0) is a solution to (2.10). We say that λ_0 is a* **bifurcation point** *if for any neighborhood U of (x_0, λ_0) there exists a $\lambda \in \Lambda$ and $x_1, x_2 \in X$, $x_1 \neq x_2$ such that $(x_1, \lambda), (x_2, \lambda) \in U$ and (x_1, λ) and (x_2, λ) are both solutions to (2.10).*

Given a solution (x_0, λ_0) to (2.10), we are interested in analyzing solutions to (2.10) in a neighborhood of (x_0, λ_0) to determine whether it is a bifurcation point. One of the most useful tools for this is the Implicit Function Theorem 10.5. This theorem asserts that if $D_x F(x_0, \lambda_0)$ is invertible, then there exists a neighborhood $U_1 \times V_1 \subset U \times V$ and a continuous function $f : V_1 \to U_1$ such that all solutions to (2.10) in $U_1 \times V_1$ are of the form $(f(\lambda), \lambda)$. Therefore in order for a bifurcation to occur at (x_0, λ), it follows that $D_x F(x_0, \lambda_0)$ must not be invertible.

2.4.1. *Liapunov-Schmidt Reduction.* The following discussion is taken from [11]. Let X, Λ and Z be Banach spaces and assume that $U \subset X$, $V \subset \Lambda$. For $\lambda = \lambda_0$, we require that the mapping $F : U \times V \to Z$ be a nonlinear Fredholm operator with respect to x; i.e. the linearization $D_x F(\cdot, \lambda_0)$ of $F(\cdot, \lambda_0) : U \to Z$ is a Fredholm operator. Assume that F also satisfies the following assumptions:

$$
\begin{aligned}
&F(x_0, \lambda_0) = 0 \quad \text{for some } (x_0, \lambda_0) \in U \times V, \\
&\dim \ker(D_x F(x_0, \lambda_0)) = \dim \ker(D_x F(x_0, \lambda_0)^*) = 1.
\end{aligned}
\tag{2.11}
$$

Given that $D_x F(x_0, \lambda_0)$ has a one-dimensional kernel, there exists a projection operator $P : X \to X_1 = \ker(D_x F(x_0, \lambda_0))$. Similarly, one has the projection operator $Q : Y \to Y_2 = \ker(D_x F(x_0, \lambda_0)^*)$. This allows us to decompose $X = X_1 \oplus X_2$ and $Y = Y_1 \oplus Y_2$ where $Y_1 = R(D_X F(x_0, \lambda_0))$. We will refer to the decomposition $X_1 \oplus X_2$ and $Y_1 \oplus Y_2$ induced by $D_x F(x_0, \lambda_0)$ as the **Liapunov decomposition**, and we see that $F(x, \lambda) = 0$ if and only if the following two equations are satisfied

$$
\begin{aligned}
&QF(x, \lambda) = 0, \\
&(I - Q)F(x, \lambda) = 0.
\end{aligned}
\tag{2.12}
$$

For any $x \in X$, we can write $x = v + w$, where $v = Px$ and $w = (I - P)x$. Define $G : U_1 \times W_1 \times V_1 \to Y_1$ by

$$
\begin{aligned}
&G(v, w, \lambda) = (I - Q)F(v + w, \lambda), \quad \text{where} \\
&U_1 \subset X_1, \quad W_1 \subset X_2, \quad V_1 \subset \mathbb{R} \quad \text{and} \\
&v_0 = Px_0 \in U_1, \quad w_0 = (I - P)x_0 \in W_1,
\end{aligned}
\tag{2.13}
$$

and U_1, W_1 are neighborhoods such that $U_1 + W_1 \subset U \subset X$.

Then the definition of $G(v, w, \lambda)$ implies that $G(v_0, w_0, \lambda_0) = 0$ and our choice of function spaces ensures that

$$
D_w G(v_0, w_0, \lambda_0) = (I - Q) D_x F(x_0, \lambda_0) : X_2 \to Y_1,
$$

is bijective. The Implicit Function Theorem then implies that there exist neighborhoods $U_2 \subset U_1, W_2 \subset W_1$ and $V_2 \subset V_1$ and a continuous function

$$
\begin{aligned}
&\psi : U_2 \times V_2 \to W_2 \quad \text{such that all solutions to } G(v, w, \lambda) = 0 \\
&\text{in } U_2 \times W_2 \times V_2 \quad \text{are of the form } G(v, \psi(v, \lambda), \lambda) = 0.
\end{aligned}
\tag{2.14}
$$

Insertion of the function $\psi(v, \lambda)$ into the second equation of (2.12) yields a finite-dimensional problem

$$
\Phi(v, \lambda) = QF(v + \psi(v, \lambda), \lambda) = 0. \tag{2.15}
$$

We observe that finding solutions (v, λ) to (2.15) is equivalent to finding solutions to $F(x, \lambda) = 0$ in a neighborhood of (x_0, λ_0). We will refer to the finite-dimensional problem (2.15) as the **Liapunov-Schmidt reduction** of (2.10).

Given that $\ker(D_x F(x_0, \lambda_0))$ is spanned by $\hat{v}_0$, then we can write $v = s\hat{v}_0 + v_0$. Substituting this into (2.15) we obtain

$$
\Phi(s, \lambda) = QF(s\hat{v}_0 + v_0 + \psi(s\hat{v}_0 + v_0, \lambda), \lambda) = 0. \tag{2.16}
$$

Using the reduction (2.16) and another application of the Implicit Function Theorem, one obtains the following theorem taken from [11], which allows us to determine a unique solution curve through the point (x_0, λ_0). We also include the proof for completeness.

Theorem 2.5. *Assume $F : U \times V \to Z$ is continuously differentiable on $U \times V \subset X \times \mathbb{R}$ and that assumptions (2.11) hold. Additionally we assume that*

$$D_\lambda F(x_0, \lambda_0) \notin R(D_x F(x_0, \lambda_0)). \tag{2.17}$$

Then there is a continuously differentiable curve through (x_0, λ_0); that is, there exists

$$\{(x(s), \lambda(s)) \mid s \in (-\delta, \delta),\ (x(0), \lambda(0)) = (x_0, \lambda_0)\}, \tag{2.18}$$

such that

$$F(x(s), \lambda(s)) = 0 \quad \text{for } s \in (-\delta, \delta), \tag{2.19}$$

and all solutions of $F(x, \lambda) = 0$ in a neighborhood of (x_0, λ_0) belong to the curve (2.18).

Proof. Let $x_0 = v_0 + w_0 = v_0 + \psi(v_0, \lambda_0)$. Differentiating (2.15) with respect to λ we obtain

$$\begin{aligned} &D_\lambda \Phi(v_0, \lambda_0) = \\ &QD_x F(x_0, \lambda_0) D_\lambda \psi(v_0, \lambda_0) + QD_\lambda F(x_0, \lambda_0) = QD_\lambda F(x_0, \lambda_0) \neq 0, \end{aligned} \tag{2.20}$$

where (2.20) is nonzero due to the extra assumption (2.17). The above expression simplifies due to the fact that that

$$D_x F(x_0, \lambda_0) D_\lambda \psi(v_0, \lambda_0) \in R(D_x F(x_0, \lambda_0)),$$

and Q is the projection onto $\ker(D_X F(x_0, \lambda_0)^*)$.

The fact that $D_\lambda \Phi(v_0, \lambda_0) \neq 0$ and that X_1, Y_2 and $\mathbb{R}$ are one-dimensional implies that we may apply the Implicit Function Theorem to $\Phi(v, \lambda)$ to conclude that there exists a continuously differentiable $\gamma : U_2 \to V_2 \subset \mathbb{R}$ such that

$$\gamma(v_0) = \lambda_0 \quad \text{and} \quad \Phi(v, \gamma(v)) = 0 \quad \text{for all} \quad v \in U_2 \subset X_1. \tag{2.21}$$

Therefore our reduced equation (2.15) becomes

$$\Phi(v, \gamma(v)) = QF(v + \psi(v, \gamma(v)), \gamma(v)) = 0, \tag{2.22}$$

where solutions to (2.22) are of the form

$$x(v) = v + \psi(v, \gamma(v)) \quad \text{and} \quad \lambda(v) = \gamma(v). \tag{2.23}$$

By writing $v = s\hat{v}_0 + v_0$ as in (2.16) and inserting this into (2.23), we obtain our solution curve

$$x(s) = v_0 + s\hat{v}_0 + \psi(v_0 + s\hat{v}_0, \gamma(v_0 + s\hat{v}_0)), \tag{2.24}$$

$$\lambda(s) = \gamma(v_0 + s\hat{v}_0). \tag{2.25}$$

□

Now we compile some useful properties of the maps $\Phi(v, \lambda)$, $\psi(v, \lambda)$ and $\gamma(v)$ defined in the (2.15), (2.21) and (2.33). These results, along with their proofs, are taken from [11].

Proposition 2.6. *Let the assumptions of Theorem 2.5 hold and let the operators $\Phi(v, \lambda)$, $\psi(v, \lambda)$ and $\gamma(v)$ be defined as in (2.15), (2.21) and (2.33) and let λ_0 and $x_0 = v_0 + w_0$ be as in the previous discussion. Then*

$$D_v \Phi(v_0, \lambda_0) = 0, \quad D_v \psi(v_0, \lambda_0) = 0, \quad \text{and} \quad D_v \gamma(v_0) = 0, \tag{2.26}$$

and each of these operators has the same order of differentiability as $F(x, \lambda)$.

Proof. The fact that $\Phi(v,\lambda)$, $\psi(v,\lambda)$ and $\gamma(v)$ all have the same order of differentiability as $F(x,\lambda)$ follows from the definition of $\Phi(v,\lambda)$ and the Implicit Function Theorem 10.5. By differentiating $(I-Q)F(v+\psi(v,\lambda),\lambda)=0$ with respect to v we obtain

$$(I-Q)D_xF(v+\psi(v,\lambda),\lambda)(I_{X_1}+D_v\psi(v,\lambda))=0, \tag{2.27}$$

where I_{X_1} denotes the identity on $X_1=\ker(D_xF(x_0,\lambda_0))$. By evaluating at (v_0,λ_0), where $x_0=v_0+w_0$, we obtain

$$(I-Q)D_xF(x_0,\lambda_0)D_v\psi(v_0,\lambda_0)=0. \tag{2.28}$$

Given that $D_v\psi(v_0,\lambda_0)$ maps onto X_2 and $(I-Q)D_XF(x_0,\lambda_0)$ is an invertible operator from X_2 to Y_1, we have that $D_v\psi(x_0,\lambda_0)=0$.

Then if we differentiate $\Phi(v,\lambda)=QF(v+\psi(v,\lambda),\lambda)=0$ with respect to v and evaluate at (v_0,λ_0), we obtain

$$D_v\Phi(v_0,\lambda_0)=QD_xF(x_0,\lambda_0)I_{X_1}=0. \tag{2.29}$$

By differentiating (2.22) with respect to v and utilizing (2.29), we have

$$D_\lambda\Phi(v_0,\lambda_0)D_v\gamma(v_0)=0.$$

The assumption that $D_\lambda\Phi(v_0,\lambda_0)\neq 0$ implies that

$$D_v\gamma(v_0)=0. \tag{2.30}$$

□

Once we've obtained a unique solution curve $(x(s),\lambda(s))$ through (x_0,λ_0), we analyze $\ddot{\lambda}(0)$ (where $\dot{}=\frac{d}{ds}$) to determine additional information about the solution curve. In particular, we can determine whether or not a **saddle node bifurcation** or fold occurs at (x_0,λ_0). This type of bifurcation occurs when the solution curve $\{x(s),\lambda(s)\}$ has a turning point at (x_0,λ_0). The next proposition, taken from [11], provides us with a method to determine information about $\ddot{\lambda}(0)$.

Proposition 2.7. *Let the assumptions of Theorem 2.5 be in effect. Additionally assume that ker$(D_XF(x_0,\lambda_0))$ is spanned by $\hat{v}_0$. Then*

$$\left.\frac{d}{ds}F(x(s),\lambda(s))\right|_{s=0}= \tag{2.31}$$
$$D_xF(x_0,\lambda_0)\dot{x}(0)+D_\lambda F(x_0,\lambda_0)\dot{\lambda}(0)=D_xF(x_0,\lambda_0)\hat{v}_0=0$$
$$\left.\frac{d^2}{ds^2}F(x(s),\lambda(s))\right|_{s=0}= \tag{2.32}$$
$$D^2_{xx}F(x_0,\lambda_0)[\hat{v}_0,\hat{v}_0]+D_xF(x_0,\lambda_0)\ddot{x}(0)+D_\lambda F(x_0,\lambda_0)\ddot{\lambda}(0)=0.$$

In particular, an application of the projection operator Q defined in (2.12) to (2.32) yields

$$QD^2_{xx}F(x_0,\lambda_0)[\hat{v}_0,\hat{v}_0]+QD_\lambda F(x_0,\lambda_0)\ddot{\lambda}(0)=0. \tag{2.33}$$

This implies that if $D_\lambda F(x_0,\lambda_0)\notin R(D_xF(x_0,\lambda_0))$ and

$$D^2_{xx}F(x_0,\lambda_0)[\hat{v}_0,\hat{v}_0]\notin R(D_xF(x_0,\lambda_0)),$$

then $\ddot{\lambda}(0)\neq 0$.

Proof. Let $\{x(s), \lambda(s)\}$ be the solution curves for $F(x,\lambda) = 0$ defined by (2.24) and (2.25). Differentiating these curves we obtain

$$\frac{d}{ds}x(s)\Big|_{s=0} = \hat{v}_0 + D_v\psi(v_0,\lambda_0)\hat{v}_0 + D_\lambda\psi(v_0,\lambda_0)D_v\gamma(v_0)\hat{v}_0 = \hat{v}_0, \tag{2.34}$$

$$\frac{d}{ds}\lambda(s)\Big|_{s=0} = D_v\gamma(v_0)\hat{v}_0 = 0, \tag{2.35}$$

where the above expressions simplify as a result of Proposition 2.6. Differentiating the expression $F(x(s),\lambda(s)) = 0$ twice and again using Proposition 2.6 to simplify, we obtain

$$\frac{d^2}{ds^2}F(x(s),\lambda(s))\Big|_{s=0} = \tag{2.36}$$
$$D^2_{xx}F(x_0,\lambda_0)[\hat{v}_0,\hat{v}_0] + D_xF(x_0,\lambda_0)\ddot{x}(0) + D_\lambda F(x_0,\lambda_0)\ddot{\lambda}(0) = 0,$$

where

$$\ddot{\lambda}(0) = D^2_{vv}\gamma(v_0)[\hat{v}_0,\hat{v}_0] \quad \text{and} \quad \ddot{x}(0) = D^2_{vv}\psi(v_0,\lambda_0)[\hat{v}_0,\hat{v}_0],$$

by differentiating (2.34) and (2.35) once more with respect to s. Applying the projection operator Q to (2.36) yields (**??**). Then the assumptions that $D_\lambda F(x_0,\lambda_0) \notin R(D_xF(x_0,\lambda_0))$ and $D^2_{xx}F(x_0,\lambda_0)[\hat{v}_0,\hat{v}_0] \notin R(D_xF(x_0,\lambda_0))$ imply that $\ddot{\lambda}(0) \neq 0$. □

The significance of Proposition 2.7 is that it gives explicit conditions that allow us to determine whether or not $\ddot{\lambda}(0)$ is nonzero. Heuristically, the fact that $\ddot{\lambda}(0) \neq 0$ means that $\lambda(s)$ has a turning point at $s = 0$. This means that the graph of $\{x(s),\lambda(s)\}$ looks like a parabola and that a saddle node bifurcation occurs at $s = 0$ (cf. [11]). If we assume that $F(x,\lambda)$ is at least 3-times differentiable we may expand the operators $\psi(v_0 + s\hat{v}_0, \gamma(v_0 + s\hat{v}_0))$ and $\gamma(v_0 + s\hat{v}_0)$ about $s = 0$ as a second order Taylor series and use (2.24) and (2.25) to obtain second order representations of our solutions $\{x(s),\lambda(s)\}$. This is the solution approach we take to prove non-uniqueness in both the CMC and non-CMC cases.

3. Main Results

The main results of this article pertain to the following one parameter family of problems

$$-\Delta\phi + a_R\phi + \lambda^2 a_\tau\phi^5 - a_{\mathbf{w}}\phi^{-7} - 2\pi\rho e^{-\lambda}\phi^5 = 0, \tag{3.1}$$
$$\mathbb{L}\mathbf{w} + \lambda b^a_\tau\phi^6 = 0.$$

Here we assume that g_{ab} is a given SPD metric with no conformal killing fields that has constant, positive scalar curvature. The expressions D_a and Δ denote the derivative and the Laplace-Beltrami operator associated with g_{ab} and

$$\mathbb{L}\mathbf{w} = -D_b(\mathcal{L}\mathbf{w})^{ab},$$

denotes the divergence of the conformal killing operator associated with g_{ab}. Finally, we define

$$a_R = \frac{1}{8}R, \qquad a_\tau = \frac{1}{12}\tau^2, \tag{3.2}$$
$$a_{\mathbf{w}} = \frac{1}{8}(\sigma + \mathcal{L}\mathbf{w})_{ab}(\sigma + \mathcal{L}\mathbf{w})^{ab}, \qquad b_\tau = \frac{2}{3}D^a\tau.$$

In general, we assume that $\tau \in C^{1,\alpha}(\mathcal{M})$, however when we prove our CMC results we will additionally require that τ be constant. For the remainder of this paper we assume that R is a positive constant and that $|\sigma| = (\sigma_{ab}\sigma^{ab})^{\frac{1}{2}}$ is also a nonzero constant. Notice that (3.1) has the form of (1.6) with initial data depending on λ where

$$\tau_\lambda = \lambda\tau, \quad \rho_\lambda = e^{-\lambda}\rho \quad \text{and} \quad \mathbf{j}_\lambda = 0.$$

We show that in both the CMC and non-CMC cases that solutions to (3.2) are non-unique. Our method for doing this is to apply the bifurcation theory outlined in Section 2.4. The first step in doing this is to formulate (3.1) in a way that allows us to utilize the framework outlined in Section 2.4.

3.1. **Problem Setup.** We now formulate (3.1) so that we can apply the Liapunov-Schmidt reduction. Define $F((\phi, \mathbf{w}), \lambda)$ by

$$F((\phi,\mathbf{w}),\lambda) = \begin{bmatrix} -\Delta\phi + a_R\phi + \lambda^2 a_\tau \phi^5 - a_{\mathbf{w}}\phi^{-7} - 2\pi\rho e^{-\lambda}\phi^5 \\ \mathbb{L}\mathbf{w} + \lambda b_\tau^a \phi^6 \end{bmatrix}, \tag{3.3}$$

and in the event that τ is constant, define

$$G(\phi,\lambda) = -\Delta\phi + a_R\phi + \lambda^2 a_\tau\phi^5 - \frac{1}{8}\sigma^2\phi^{-7} - 2\pi\rho e^{-\lambda}\phi^5. \tag{3.4}$$

If $F((\phi, \mathbf{w}), \lambda) = 0$ (resp. $G(\phi, \lambda) = 0$) for a given λ, then $((\phi, \mathbf{w}), \lambda)$ (resp. (ϕ, λ)) solves Eq. (3.1) (resp. Eq. (3.4)).

We view (3.3) and (3.4) as nonlinear operators between the Banach spaces

$$F((\phi,\mathbf{w}),\lambda) : C^{k,\alpha}(\mathcal{M}) \oplus C^{k,\alpha}(\mathcal{TM}) \times \mathbb{R} \to C^{k-2,\alpha}(\mathcal{M}) \oplus C^{k-2,\alpha}(\mathcal{TM}),$$

$$G(\phi,\lambda) : C^{k,\alpha}(\mathcal{M}) \times \mathbb{R} \to C^{k-2,\alpha}(\mathcal{M}).$$

where $k \geq 2$. For $\phi \neq 0$ and $X = (\phi, \mathbf{w})$, the first order Fréchet derivatives $D_\phi G(\phi, \lambda)$, $D_\lambda G(\phi, \lambda)$, $D_X F((\phi, \mathbf{w}), \lambda)$ and $D_\lambda F((\phi, \mathbf{w}), \lambda)$ all exist. In fact, both F and G are k-differentiable for any $k \in \mathbb{N}$ provided that $\phi \neq 0$. See the Appendix 10.1 for more information regarding Fréchet derivatives.

Now we are ready to state the main results of this paper. The first two results state that there is a critical density $\rho = \rho_c$ such that there exists a constant ϕ_c where the linearizations $D_\phi G(\phi_c, 0)$ and $D_X F((\phi_c, \mathbf{0}), 0)$ have a kernel of dimension one. This provides the basis for our final two main results where we determine explicit solution curves $\{\phi(s), \lambda(s)\}$ and $\{(\phi(s), \mathbf{w}(s)), \lambda(s)\}$ to obtain our non-uniqueness results.

3.2. **Existence of ρ_c such that dim ker$(D_X F((\phi_c, \mathbf{0}, 0)) = 1$.** The two results in this section pertain to the existence of a critical energy density $\rho = \rho_c$ at which the linearizations of the operators F and G develop a one-dimensional kernel. These results allow us to apply the Liapunov-Schmidt reduction outlined in Section 2.4 to analyze solutions in a neighborhood of $((\phi_c, \mathbf{0}), 0)$ and $(\phi_c, 0)$. We present the theorems here without proof and postpone them until Section 5.

Theorem 3.1 (CMC). *Let $D_\phi G(\phi, \lambda)$ denote the Fréchet derivative of* (3.4) *with respect to ϕ. Then there exists a critical value of $\rho = \rho_c$ and a constant ϕ_c such that when $\rho = \rho_c$, Eq.* (3.4) *has a solution if and only if $\lambda \geq 0$. Furthermore, dim ker$(D_\phi G(\phi_c, 0))) = 1$ and it is spanned by the constant function $\phi = 1$. Moreover, we can determine the explicit values of ρ_c and ϕ_c, which are*

$$\rho_c = \frac{R^{\frac{3}{2}}}{24\sqrt{3}\pi|\sigma|} \quad \textit{and} \quad \phi_c = \left(\frac{R}{24\pi\rho}\right)^{\frac{1}{4}}. \tag{3.5}$$

Proof. We present the proof in Section 5. □

Theorem 3.2 (non-CMC)**.** *Let $D_XF((\phi,\mathbf{w}),\lambda)$ denote the Fréchet derivative of Eq.* (3.3) *with respect to $X = (\phi,\mathbf{w})$ and let ρ_c and ϕ_c be as in Theorem 3.1. Then when $\rho = \rho_c$, dim ker$(D_XF((\phi_c,\mathbf{0}),0))) = 1$ and it is spanned by the constant vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.*

Proof. We present the proof in Section 5. □

3.3. **Non-unique Solutions to $F((\phi,\mathbf{w}),\lambda) = 0$ when $\rho = \rho_c$.** The two Theorems in this section pertain to the non-uniqueness of solutions to the nonlinear problems (3.3) and (3.4). Theorem 3.3 provides the explicit form of solutions to (3.4) in a neighborhood of the point $(\phi_c, 0)$ in the CMC case. The form of this solution curve implies that a saddle node bifurcation occurs at $(\phi_c, 0)$ and that solutions are non-unique in a neighborhood of this point. Theorem 3.4 provides analogous results in the non-CMC case for the point $((\phi_c,\mathbf{0}),0)$.

Theorem 3.3 (CMC)**.** *Suppose that τ is constant. Then* (3.3) *reduces to the scalar problem*

$$-\Delta\phi + a_R\phi + (\lambda^2 a_\tau - 2\pi\rho e^{-\lambda})\phi^5 - \frac{1}{8}\sigma^2\phi^{-7} = 0. \tag{3.6}$$

When $\rho = \rho_c$, with ρ_c as in Theorem 3.2, then there exists a neighborhood of $(\phi_c, 0)$ such that all solutions to (3.6) *in this neighborhood lie on a smooth solution curve $\{\phi(s),\lambda(s)\}$ that has the form*

$$\phi(s) = \phi_c + s + O(s^2), \tag{3.7}$$

$$\lambda(s) = \frac{1}{2}\ddot{\lambda}(0)s^2 + O(s^3), \quad (\ddot{\lambda}(0) \neq 0). \tag{3.8}$$

In particular, there exists a $\delta > 0$ such that for all $0 < \lambda < \delta$ there exist at least two distinct solutions $\phi_{1,\lambda} \neq \phi_{2,\lambda}$ to (3.6)*.*

Proof. We postpone the proof until Section 7. □

Theorem 3.4 (non-CMC)**.** *Suppose $\tau \in C^{1,\alpha}(\mathcal{M})$ is non-constant and let $F((\phi,\mathbf{w}),\lambda)$ be defined as in* (3.3)*. Then if ρ_c and ϕ_c are defined as in Theorem 3.1 and $\rho = \rho_c$, there exists a neighborhood of $((\phi_c,\mathbf{w}),0)$ such that all solutions to $F((\phi,\mathbf{w}),\lambda) = 0$ in this neighborhood lie on a smooth curve of the form*

$$\phi(s) = \phi_c + s + \frac{1}{2}\ddot{\lambda}(0)u(x)s^2 + O(s^3), \tag{3.9}$$

$$\mathbf{w}(s) = \frac{1}{2}\ddot{\lambda}(0)\mathbf{v}(x)s^2 + O(s^3),$$

$$\lambda(s) = \frac{1}{2}\ddot{\lambda}(0)s^2 + O(s^3), \quad (\ddot{\lambda}(0) \neq 0),$$

where $u(x) \in C^{2,\alpha}(\mathcal{M})$, $\mathbf{v}(x) \in C^{2,\alpha}(\mathcal{TM})$ and $\mathbf{v}(x) \neq \mathbf{0}$. In particular, there exists a $\delta > 0$ such that for all $0 < \lambda < \delta$ there exist elements $(\phi_{1,\lambda},\mathbf{w}_{1,\lambda}), (\phi_{2,\lambda},\mathbf{w}_{2,\lambda}) \in C^{2,\alpha}(\mathcal{M}) \oplus C^{2,\alpha}(\mathcal{TM})$ such that

$$F((\phi_{i,\lambda},\mathbf{w}_{i,\lambda}),\lambda) = 0, \text{ for } i \in \{1,2\}, \text{ and } (\phi_{1,\lambda},\mathbf{w}_{1,\lambda}) \neq (\phi_{2,\lambda},\mathbf{w}_{2,\lambda}).$$

Proof. We present the proof in Section 8. □

4. Some Key Technical Results

4.1. **Existence of a Critical Value ρ_c.** In this section we lay the foundation for proving Theorems 3.1 and 3.2. As in [14], we seek a critical density ρ_c where our elliptic problem goes from having positive solutions to having no positive solutions. In particular, what we seek is a value ρ_c such that when $\lambda = 0$, then (3.3) will have no solution for $\rho > \rho_c$ and will have a solution for $\rho \leq \rho_c$.

When $\lambda = 0$, the assumption that g_{ab} admits no conformal killing fields implies that

$$F((\phi,\mathbf{w}),0) = F((\phi,\mathbf{0}),0) = \begin{bmatrix} -\Delta\phi + a_R\phi - \frac{\sigma^2}{8}\phi^{-7} - 2\pi\rho\phi^5 = 0 \\ \mathbf{w} = 0 \end{bmatrix}. \tag{4.1}$$

Define

$$q(\chi) = a_R\chi - \frac{1}{8}\sigma^2\chi^{-7} - 2\pi\rho_c\chi^5, \tag{4.2}$$

where ρ_c is a constant to be determined. The objective will be to determine ρ_c so that $q(\chi)$ has a single, positive, multiple root and then use the maximum principle discussed in Appendix 10.6 to conclude that if $\rho > \rho_c$, then (4.1) will have no solution. This leads us to the following proposition.

Proposition 4.1. *Let $q(\chi)$ be defined as in (4.2). Then there exists constants $\rho_c > 0$ and $\phi_c > 0$ such that $q(\chi) \leq 0$ for all $\chi > 0$ and the only positive root of $q(\chi)$ is ϕ_c.*

Proof. To determine ρ_c, we observe that because a_R and σ^2 are constants, we simply need to analyze the roots of (4.2) as ρ_c varies. We seek ρ_c such that $q(\chi)$ has a single, positive, multiple root. We observe that $q(\chi) = 0$ if and only if

$$p(\chi) = a_R\chi^8 - \frac{1}{8}\sigma^2 - 2\pi\rho_c\chi^{12} = 0.$$

Furthermore, it is clear that each pair of roots $\{-\chi_0, \chi_0\}$ of the even polynomial $p(\chi)$ is in direct correspondence with each positive root of $p(\gamma) = a_R\gamma^2 - \frac{1}{8}\sigma^2 - 2\pi\rho_c\gamma^3$, where $\gamma = \chi^4$. Therefore, we simply need to choose ρ_c such that $p(\gamma)$ has a single positive root. To accomplish this, we find the lone, local maximum of $p(\gamma)$ and require it to be a root of $p(\gamma)$. We have that

$$0 = p'(\gamma) = 2a_R\gamma - 6\pi\rho_c\gamma^2 \Longrightarrow \gamma_c = \frac{a_R}{3\pi\rho_c} \quad \text{is a local max},$$

and

$$\begin{aligned} 0 = p(\gamma_c) &= a_R\left(\frac{a_R}{3\pi\rho_c}\right)^2 - \frac{1}{8}\sigma^2 - 2\pi\rho_c\left(\frac{a_R}{3\pi\rho_c}\right) \\ &= \frac{a_R^3 - \frac{1}{8}\sigma^2(27\pi^2\rho_c^2)}{27\pi^2\rho_c^2} \Longrightarrow \rho_c = \frac{R^{\frac{3}{2}}}{24\sqrt{3}|\sigma|\pi}. \end{aligned} \tag{4.3}$$

□

The next result follows immediately from the previous analysis but will be useful going forward.

Corollary 4.2. *Define the constants*

$$\rho_c = \frac{R^{\frac{3}{2}}}{24\sqrt{3}|\sigma|\pi} \quad \textit{and} \quad \phi_c = \left(\frac{a_R}{3\pi\rho_c}\right)^{\frac{1}{4}}. \tag{4.4}$$

Then if

$$q(\chi) = a_R\chi - \frac{1}{8}\sigma^2\chi^{-7} - 2\pi\rho_c\chi^5,$$

it follows that $q(\phi_c) = q'(\phi_c) = 0$.

Proof. This follows immediately from the proof of Proposition 4.1 or by direct computation. □

Now we show that ρ_c is a critical value of (4.1).

Proposition 4.3. *Let* $\rho(x) \in C(\mathcal{M})$. *Then the constant* ρ_c *defined in Corollary 4.2 has the property that Eq. (4.1) has a positive solution if* $0 < \rho \leq \rho_c$ *and has no positive solution if* $\rho > \rho_c$.

Proof. Let $q(\chi)$ be defined as in Corollary 4.2. If $\phi > 0$ solves (4.1), then

$$\Delta\phi = a_R\phi - \frac{1}{8}\sigma^2\phi^{-7} - 2\pi\rho\phi^5 = f(x,\phi). \tag{4.5}$$

We observe that if $\rho > \rho_c$, then $\check{\rho} = \inf_{x\in\mathcal{M}}\rho > \rho_c$ and for $\chi > 0$,

$$f(x,\chi) = a_R\chi - \frac{1}{8}\sigma^2\chi^{-7} - 2\pi\rho\chi^5 \leq a_R\chi - \frac{1}{8}\sigma^2\chi^{-7} - 2\pi\check{\rho}\chi^5 < q(\chi). \tag{4.6}$$

Therefore if $\rho > \rho_c$, (4.5) and (4.6) imply that any positive solution ϕ to (4.1) satisfies

$$\Delta\phi = f(x,\phi) < q(\phi) \leq 0.$$

An application of the maximum principle (10.6) implies that if $\rho > \rho_c$, then (4.1) has no solution.

To verify that (4.1) has a solution if $\rho \leq \rho_c$, first observe that Corollary 4.2 implies that

$$\phi_c = \left(\frac{a_R}{3\pi\rho_c}\right)^{\frac{1}{4}} = \left(\frac{R}{24\pi\rho_c}\right)^{\frac{1}{4}}, \tag{4.7}$$

solves Eq. (4.1) when $\rho = \rho_c$. If $\rho < \rho_c$, the properties of $q(\chi)$ imply that the polynomial

$$q_1(\chi) = a_R\chi - \frac{1}{8}\sigma^2\chi^{-7} - 2\pi\hat{\rho}\chi^5, \qquad \hat{\rho} = \sup_{x\in\mathcal{M}}\rho(x),$$

will have two positive roots $\chi_1 < \chi_2$. Therefore, any ϕ_+ satisfying $0 < \chi_1 < \phi_+ < \chi_2$ will be a positive super-solution to (4.1) given that

$$f(x,\chi) > q_1(\chi) = a_R\chi - \frac{1}{8}\sigma^2\chi^{-7} - 2\pi\hat{\rho}\chi^5.$$

Similarly, we may choose a positive sub-solution $\phi_- < \phi_+$ to (4.1) by choosing any sufficiently small ϕ_- satisfying $0 < \phi_- < \chi_3$, where χ_3 is the lone positive root of

$$q_2(\chi) = a_R\chi - \frac{1}{8}\sigma^2\chi^{-7}.$$

We can then apply the method of sub- and super-solutions outlined in Section 10.2.2 to solve (4.1). □

The next result extends Proposition 4.3 to the case when $\lambda \neq 0$ and indicates that ρ_c is also a critical value for the decoupled problem (3.4).

Corollary 4.4. *Let $\rho(x) \in C(\mathcal{M})$ and suppose that τ is a constant and that*

$$\rho_c = \frac{R^{\frac{3}{2}}}{24\sqrt{3}|\sigma|\pi}.$$

There exists an $\epsilon > 0$ such that there is no positive solution to (3.4) if $\rho > \rho_c$ and $-\epsilon < \lambda < 0$, and there exists a positive solution to (3.4) if $0 < \rho \leq \rho_c$ and $0 \leq \lambda < \epsilon$. Finally, if $\rho = \rho_c$ and λ is sufficiently small, then (3.4) has a solution if and only if $\lambda \geq 0$.

Proof. Again, we observe that if $\phi > 0$ solves (3.4), then

$$\Delta\phi = a_R\phi + \lambda^2 a_\tau \phi^5 - \frac{1}{8}\sigma^2\phi^{-7} - 2\pi\rho e^{-\lambda}\phi^5 = f(x,\phi,\lambda). \tag{4.8}$$

Let $q(\chi)$ be as in Corollary 4.2 and define

$$p_1(\chi,\lambda) = a_R\chi + \lambda^2 a_\tau\chi^5 - \frac{1}{8}\sigma^2\chi^{-7} - 2\pi\check{\rho}e^{-\lambda}\chi^5,$$

where $\check{\rho} = \inf_{x\in\mathcal{M}}\rho(x)$. It is clear that $f(x,\phi,\lambda) \leq p_1(\phi,\lambda)$ for any $\phi > 0$, and for $\lambda < 0$ and $\rho > \rho_c$ we have that

$$\begin{aligned} p_1(\chi,\lambda) &= a_R\chi + \lambda^2 a_\tau\chi^5 - \frac{1}{8}\sigma^2\chi^{-7} - 2\pi\check{\rho}e^{-\lambda}\chi^5 \\ &\leq a_R\chi + (\lambda^2 a_\tau - 2\pi\rho_c + 2\pi\rho_c\lambda + o(\lambda^2))\chi^5 - \frac{1}{8}\sigma^2\chi^{-7} \\ &= q(\chi) + (\lambda^2 a_\tau + 2\pi\rho_c\lambda + o(\lambda^2))\chi^5 = q(\chi) + g(\lambda)\chi^5. \end{aligned} \tag{4.9}$$

Here we observe that $g(\lambda) \to 0$ as $\lambda \to 0$, and for $|\lambda|$ sufficiently small, $g(\lambda) < 0$ if $\lambda < 0$. By Proposition 4.1, we know that if $\chi > 0$ then $q(\chi) \leq 0$. So Eq. (4.9) implies that if $\rho > \rho_c$ and $\lambda < 0$ is sufficiently small, then $f(x,\chi,\lambda) \leq p_1(\chi,\lambda) < 0$, and the maximum principle then implies that (3.4) will have no solution.

If $\rho \leq \rho_c$, then define

$$p_2(\chi,\lambda) = a_R\chi + \lambda^2 a_\tau\chi^5 - \frac{1}{8}\sigma^2\chi^{-7} - 2\pi\hat{\rho}e^{-\lambda}\chi^5,$$

where $\hat{\rho} = \sup_{x\in\mathcal{M}}\rho(x)$. It is clear that $f(x,\chi,\lambda) \geq p_2(\chi,\lambda)$ for all $\chi > 0$, and for $\lambda \leq 0$ we have

$$\begin{aligned} p_2(\chi,\lambda) &= a_R\chi + \lambda^2 a_\tau\chi^5 - \frac{1}{8}\sigma^2\chi^{-7} - 2\pi\hat{\rho}e^{-\lambda}\chi^5 \\ &\geq a_R\chi + (\lambda^2 a_\tau - 2\pi\rho_c + 2\pi\rho_c\lambda + o(\lambda^2))\chi^5 - \frac{1}{8}\sigma^2\chi^{-7} \\ &= q(\chi) + (\lambda^2 a_\tau + 2\pi\rho_c\lambda + o(\lambda^2))\chi^5 = q(\chi) + g(\lambda)\chi^5. \end{aligned} \tag{4.10}$$

Again, $g(\lambda) \to 0$ as $\lambda \to 0$ and $g(\lambda) > 0$ for $\lambda > 0$ sufficiently small. Therefore if $\chi > 0$, Eq. (4.10) implies that $f(x,\chi,\lambda) > p_2(\chi,\lambda) \geq q(\chi)$ if $\lambda \geq 0$. The properties of $q(\chi)$ specified in Proposition 4.1 imply that that for any $\lambda > 0$, either $p_2(\chi,\lambda)$ has a single positive root χ_0 and $p_2(\chi,\lambda) > 0$ for all $\chi > \chi_0$, or $p_2(\chi,\lambda)$ has two distinct positive roots. This implies that if $\lambda > 0$ we can find a positive super-solution ϕ_+ to (3.4). If $\lambda = 0$ we take $\phi_+ = \phi_c$ to be a super-solution where ϕ_c is defined in Corollary 4.2. Similarly, we can also find a positive sub-solution ϕ_- satisfying $\phi_- < \phi_+$ by choosing any sufficiently small $0 < \phi_- < \chi_0$, where χ_0 is the unique positive root of

$$r(\chi,\lambda) = a_R\chi + \lambda^2 a_\tau\chi^5 - \frac{1}{8}\sigma^2\chi^{-7}.$$

The method of sub-and super-solutions outlined in Section 10.2.2 then implies that if $\rho \le \rho_c$ and $\lambda \ge 0$, then (3.4) has a solution.

Finally, we observe that if $\rho = \rho_c$, then we have that

$$f(x,\chi,\lambda) = q(\chi) + g(\lambda)\chi^5,$$

where f and g are the same as above. Therefore, when λ is small and $\rho = \rho_c$, we can apply the above analysis to conclude that (3.4) will have a solution if and only if $\lambda \ge 0$. □

Remark 4.5. *We note that the negative sign in front of the term $2\pi\rho\chi^5$ in the polynomial*

$$q(\chi) = a_R\chi - \frac{1}{8}\sigma^2\chi^{-7} - 2\pi\rho\chi^5,$$

played an essential role in allowing us to determine our critical density ρ_c and critical solution ϕ_c. If this term were positive, then $q(\chi)$ would be monotonic increasing for $\chi > 0$, and we would not be able to find a positive ϕ_c and ρ_c so that $q(\phi_c) = 0$ and $q'(\phi_c) = 0$. As we saw in Corollary 4.4 and Proposition 4.3, these properties of $q(\chi)$ played an important role in the existence of solutions to Eq. (3.4) and Eq. (4.1). Later in this article, we will also see that these properties of $q(\chi)$ play an important role in our non-uniqueness analysis by allowing for the kernel of the linearization of $F((\phi,\mathbf{w}),\lambda)$ and $G(\phi,\lambda)$ to be one-dimensional. These facts further emphasize the role that terms with the "wrong sign" (cf. [14]) have in the non-uniqueness phenomena associated with the CTS, CTT and XCTS formulations of the Einstein constraint equations.

4.2. **Existence of a One Dimensional kernel of $D_XF((\phi_c,\mathbf{0}),0)$ when $\rho=\rho_c$.** In the previous section we proved the existence of a critical density ρ_c that affected whether Eq. (4.1) and Eq. (3.4) had positive solutions. We now show that when $\rho = \rho_c$, the linearization of both (3.4) and (3.3) develops a one-dimensional kernel.

We first calculate the Fréchet derivatives $D_XF((\phi,\mathbf{w}),\lambda)$ and $D_\phi G(\phi,\lambda)$. To compute these derivatives, we need only compute the Gâteaux derivatives given that the G-derivatives are continuous in a neighborhood of $((\phi_c,\mathbf{0},0)$. See [16] and Remark 10.2. Therefore,

$$D_XF((\phi_c,\mathbf{0}),0) = \frac{d}{dt}F((\phi_c+t\phi,t\mathbf{w}),0)\bigg|_{t=0},$$

where $(\phi,\mathbf{w}) \in C^{2,\alpha}(\mathcal{M}) \oplus C^{2,\alpha}(\mathcal{TM})$ satisfies $\|(\phi,\mathbf{w})\|_{C^{2,\alpha}(\mathcal{M})\oplus C^{2,\alpha}(\mathcal{TM})} = 1$.

So for a given $((\phi,\mathbf{w}),\lambda)$, the Fréchet derivative

$$D_XF((\phi,\mathbf{w}),\lambda) : C^{2,\alpha}(\mathcal{M}) \oplus C^{2,\alpha}(\mathcal{TM}) \to C^{0,\alpha}(\mathcal{M}) \oplus C^{0,\alpha}(\mathcal{TM}),$$

is a block matrix of operators where the first column consists of derivatives of $F((\phi,\mathbf{w}),\lambda)$ with respect to ϕ and the second column consists of derivatives with respect to $\mathbf{w}$. This implies that

$$D_XF((\phi,\mathbf{w}),\lambda) = \begin{bmatrix} -\Delta + a_R + 5\lambda^2 a_\tau\phi^4 + 7a_{\mathbf{w}}\phi^{-8} - 10\pi\rho_c e^{-\lambda}\phi^4 & \overline{\mathbb{L}} \\ 6\lambda b_\tau^a\phi^5 & \mathbb{L} \end{bmatrix}, \quad (4.11)$$

where

$$\overline{\mathbb{L}}h = \overline{\mathbb{L}}(\phi,\mathbf{w})h = -\frac{1}{4}\phi^{-7}\left((\mathcal{L}w)_{ab}(\mathcal{L}h)^{ab} + \sigma_{ab}(\mathcal{L}h)^{ab}\right), \quad (4.12)$$

and $\mathcal{L}$ is the conformal Killing operator. Similarly, in the CMC case the map

$$D_\phi G(\phi,\lambda) : C^{2,\alpha}(\mathcal{M}) \to C^{0,\alpha}(\mathcal{M}),$$

has the form

$$D_\phi G(\phi,\lambda) = -\Delta + a_R + 5\lambda^2 a_\tau \phi^4 + \frac{7}{8}\sigma^2\phi^{-8} - 10\pi\rho_c e^{-\lambda}\phi^4. \tag{4.13}$$

We now make some key observations about (5.1).

Proposition 4.6. *Let ϕ_c be as in Corollary 4.2. Then $F((\phi_c,\mathbf{0}),0)=0$ and $D_X F((\phi_c,\mathbf{0}),0)$ has the form*

$$D_X F(\phi_c,\mathbf{0},0) = \begin{bmatrix} -\Delta & \tilde{\mathbb{L}} \\ 0 & \mathbb{L} \end{bmatrix}, \tag{4.14}$$

where $\tilde{\mathbb{L}} : C^{k,\alpha}(\mathcal{TM}) \to C^{k-1,\alpha}(\mathcal{M})$ is defined by

$$\overline{\mathbb{L}}(\phi_c,\mathbf{0})h = \tilde{\mathbb{L}}h = -\frac{1}{4}\phi_c^{-7}\sigma_{ab}(\mathcal{L}h)^{ab},$$

and $\mathcal{L}$ is the conformal killing operator.

Proof. By Corollary 4.2 it follows that ϕ_c is a root of the polynomial

$$q(\chi) = a_R\chi - \frac{1}{8}\sigma^2\chi^{-7} - 2\pi\rho_c\chi^5,$$

and also a root of

$$q'(\chi) = a_R + \frac{7}{8}\sigma^2\chi^{-8} - 10\pi\rho_c\chi^4. \tag{4.15}$$

This implies that $F((\phi_c,\mathbf{0}),0)=0$ and that Eq. (5.1) reduces to (5.4) when $((\phi,\mathbf{w}),\lambda) = ((\phi_c,\mathbf{0}),0)$. □

Remark 4.7. *Corollary 4.2 implies that (5.3) reduces to*

$$D_\phi G(\phi_c,0) = -\Delta, \tag{4.16}$$

in the CMC case. Therefore $\dim \ker(D_\phi G(\phi_c,0)))=1$ and it is spanned by the constant function $\phi=1$.

Corollary 4.8. *Letting $\mathcal{H}_1 = L^2(\mathcal{M})$ and $\mathcal{H}_2 = L^2(\mathcal{TM})$, the $\mathcal{H}_1\oplus\mathcal{H}_2$-adjoint of $D_X F((\phi_c,\mathbf{0}),0)$ has the form*

$$(D_X F(\phi_c,\mathbf{0},0))^* = \begin{bmatrix} -\Delta & 0 \\ \hat{\mathbb{L}} & \mathbb{L} \end{bmatrix}, \tag{4.17}$$

where $\hat{\mathbb{L}} : C^{k,\alpha}(\mathcal{M}) \to C^{k-1,\alpha}(\mathcal{TM})$ is defined by

$$\hat{\mathbb{L}}u = D^b(\frac{1}{4}\phi_c^{-7}u\sigma_{ab}). \tag{4.18}$$

Proof. Let $(u_1,\mathbf{v}_1)$ and $(u_2,\mathbf{v}_2)$ both be elements of $C^2(\mathcal{M})\oplus C^2(\mathcal{TM})$. Then given that both $-\Delta$ and $\mathbb{L} = -D_b(\mathcal{L})^{ab}$ are self-adjoint with respect to the $L^2(\mathcal{M})$ and $L^2(\mathcal{TM})$ inner products, it follows that

$$\left\langle D_X F((\phi_c,\mathbf{0}),0)\begin{bmatrix} u_1 \\ \mathbf{v}_1 \end{bmatrix}, \begin{bmatrix} u_2 \\ \mathbf{v}_2 \end{bmatrix}\right\rangle = \int_{\mathcal{M}} (-u_1\Delta u_2 + \mathbf{v}_1\cdot\mathbb{L}\mathbf{v}_2 + \tilde{\mathbb{L}}\mathbf{v}_1 u_2)dV_g, \tag{4.19}$$

where dV_g is the volume element associated with g_{ab} and $\tilde{\mathbb{L}}\mathbf{v}_1 = -\frac{1}{4}\phi_c^{-7}\sigma_{ab}(\mathcal{L}\mathbf{v}_1)^{ab}$. Given that the negative divergence of a $(0,2)$ tensor and the conformal killing operator $\mathcal{L}$ are formal adjoints (see [16]), we have that

$$\int_{\mathcal{M}} \tilde{\mathbb{L}}\mathbf{v}_1 u_2 dV_g = \int_{\mathcal{M}} \left(-\frac{1}{4}u_2\phi_c^{-7}\sigma_{ab}(\mathcal{L}\mathbf{v}_1)^{ab}\right) dV_g \tag{4.20}$$

$$= \int_{\mathcal{M}} \left(D^b(\frac{1}{4}u_2\phi_c^{-7}\sigma_{ab})\cdot \mathbf{v}_1\right) dV_g = \int_{\mathcal{M}} \hat{\mathbb{L}}u_2 \cdot \mathbf{v}_1 dV_g.$$

Therefore,

$$\left\langle D_X F((\phi_c,\mathbf{0}),0)\begin{bmatrix} u_1 \\ \mathbf{v}_1 \end{bmatrix}, \begin{bmatrix} u_2 \\ \mathbf{v}_2 \end{bmatrix}\right\rangle = \tag{4.21}$$

$$\int_{\mathcal{M}} (-u_1\Delta u_2 + \mathbf{v}_1\cdot \mathbb{L}\mathbf{v}_2 + \hat{\mathbb{L}}u_2\cdot \mathbf{v}_1)dV_g = \left\langle \begin{bmatrix} u_1 \\ \mathbf{v}_1 \end{bmatrix}, \begin{bmatrix} -\Delta & 0 \\ \hat{\mathbb{L}} & \mathbb{L} \end{bmatrix}\begin{bmatrix} u_2 \\ \mathbf{v}_2 \end{bmatrix}\right\rangle.$$

□

Corollary 4.9. *$D_X F((\phi_c,\mathbf{0}),0)$ has a kernel of dimension 1 that is spanned by $\begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix}$, and $(D_X F(\phi_c,\mathbf{0},0))^*$ also has a kernel of dimension one that is spanned by $\begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix}$.*

Proof. We solve for $\begin{bmatrix} u \\ \mathbf{v} \end{bmatrix} \in C^{2,\alpha}(\mathcal{M}) \oplus C^{2,\alpha}(\mathcal{TM})$ such that

$$D_X F((\phi_c,\mathbf{0}),0)\begin{bmatrix} u \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} -\Delta & \tilde{\mathbb{L}} \\ 0 & \mathbb{L} \end{bmatrix}\begin{bmatrix} u \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{0} \end{bmatrix}.$$

Given that g_{ab} admits no conformal killing fields, we must have that $\mathbf{v} = 0$. This implies that

$$0 = -\Delta u - \frac{1}{4}\phi_c^{-7}(\sigma_{ab}(\mathcal{L}\mathbf{v})^{ab}) = -\Delta u \Longrightarrow u \quad \text{is a constant.}$$

Therefore $\begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix}$ spans $\ker(D_X F((\phi_c,\mathbf{0}),0)$.

Similarly, we solve for $\begin{bmatrix} u \\ \mathbf{v} \end{bmatrix}$ such that

$$(D_X F((\phi_c,\mathbf{0}),0))^*\begin{bmatrix} u \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} -\Delta & 0 \\ \hat{\mathbb{L}} & \mathbb{L} \end{bmatrix}\begin{bmatrix} u \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{0} \end{bmatrix}.$$

This implies that u is a constant and that

$$0 = \hat{\mathbb{L}}u + \mathbb{L}\mathbf{v} = \nabla^b(\frac{1}{4}\phi_c u\sigma_{ab}) + \mathbb{L}\mathbf{v} = \frac{1}{4}\phi_c u\nabla^b\sigma_{ab} + \mathbb{L}\mathbf{v}.$$

Given that σ_{ab} is divergence free, we have that $\nabla^b\sigma_{ab} = 0$, which implies that $\mathbf{v} = 0$. Therefore $\begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix}$ spans $\ker(D_X F((\phi_c,\mathbf{0}),0)^*)$. □

We can now prove Theorems 3.1 and 3.2. The proofs are an immediate consequence of the preceding results, but we summarize them here in the proof for convenience.

4.3. Proofs of Theorems 3.1 and 3.2: Critical Parameter and Kernel Dimension.

Proposition 4.3 implies the existence of critical values

$$\rho_c = \left(\frac{R}{24\pi\rho_c}\right)^{\frac{1}{4}} \quad \text{and} \quad \phi_c = \left(\frac{a_R}{3\pi\rho_c}\right)^{\frac{1}{4}},$$

such that if

$$q(\chi) = a_R\chi - \frac{1}{8}\sigma^2\chi^{-7} - 2\pi\rho_c\chi^5,$$

then $q(\phi_c) = q'(\phi_c) = 0$. By Remark 5.2 we have that the linearization (5.3) in the CMC case reduces to $-\Delta$. This proves Theorem 3.1. Similarly, in Proposition 5.1 we explicitly determined $D_XF((\phi_c, \mathbf{0}), 0)$, and in Corollary 4.2 we showed that it has a kernel spanned by the constant vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. This proves Theorem 3.2.

4.4. **Fredholm properties of the operators $D_XF((\phi_c, \mathbf{0}), 0)$ and $D_\phi G(\phi_c, 0)$.** Now that we have shown that the linearizations $D_XF((\phi_c, \mathbf{0}), 0)$ and $D_\phi G(\phi_c, 0)$ have one-dimensional kernels, we are almost ready to apply the Liapunov-Schmidt reduction. Recall from section 2 that a key assumption in this reduction was that the operator be a nonlinear Fredholm operator. Therefore, to apply this reduction in the CMC and non-CMC cases we must show that the operators $D_\phi G(\phi_c, 0)$ and $D_XF((\phi_c, \mathbf{0}), 0)$ are Fredholm operators between the spaces on which they are defined. In particular, we need to show that $D_\phi G(\phi_c, 0)$ is a Fredholm operator between the spaces $C^{2,\alpha}(\mathcal{M})$ and $C^{0,\alpha}(\mathcal{M})$ and that the operator $D_XF((\phi_c, \mathbf{0}), 0)$ is a Fredholm operator between $C^{2,\alpha}(\mathcal{M})\oplus C^{2,\alpha}(\mathcal{TM})$ and $C^{0,\alpha}(\mathcal{M}) \oplus C^{0,\alpha}(\mathcal{TM})$.

In the CMC case, we have that $D_\phi G(\phi_c, 0) = -\Delta$. It is well known that this operator is a Fredholm operator between the Hilbert spaces $H^2(\mathcal{M})$ and $L^2(\mathcal{M})$ [8]. Furthermore, $-\Delta$ is a Fredholm operator between the subspaces $C^{2,\alpha}(\mathcal{M})$ and $C^{0,\alpha}(\mathcal{M})$ because of the regularity properties of the the Laplacian and the fact that these spaces continuously embed into the Hilbert spaces $H^2(\mathcal{M})$ and $L^2(\mathcal{M})$. See Appendix 10.2.3 for a more detailed discussion of these facts.

Letting $L = -\Delta$, we regard $L = L^*$ as operators from $H^2(\mathcal{M}) \to L^2(\mathcal{M})$. The Fredholm properties of these operators allow us to make the following decompositions that are orthogonal with respect to the L^2-inner product:

$$\begin{aligned} L^2(\mathcal{M}) &= R(L^*) \oplus \text{ker}(L) \\ L^2(\mathcal{M}) &= R(L) \oplus \text{ker}(L^*). \end{aligned} \tag{4.22}$$

In this case, these decompositions are the same given that L is self-adjoint. Therefore if we regard $C^{2,\alpha}(\mathcal{M})$ and $C^{0,\alpha}(\mathcal{M})$ as subspaces of $L^2(\mathcal{M})$, then we may use (6.1) to obtain the following decompositions

$$\begin{aligned} C^{2,\alpha}(\mathcal{M}) &= (R(L^*) \cap C^{2,\alpha}(\mathcal{M})) \oplus \text{ker}(L), \\ C^{0,\alpha}(\mathcal{M}) &= (R(L) \cap C^{0,\alpha}(\mathcal{M})) \oplus \text{ker}(L^*), \end{aligned} \tag{4.23}$$

which are also orthogonal with respect to the L^2-inner product. See Appendix 10.2.3 for further details.

It is not as clear that the operator $D_XF((\phi_c, \mathbf{0}), 0)$ is a Fredholm operator between the spaces $C^{2,\alpha}(\mathcal{M})\oplus C^{2,\alpha}(\mathcal{TM})$ and $C^{0,\alpha}(\mathcal{M})\oplus C^{0,\alpha}(\mathcal{TM})$. For the sake of completeness, we briefly discuss this point. As in Appendix 10.2.3, we first show that $D_XF(\phi_c, \mathbf{0}), 0)$ is a Fredholm operator from the Hilbert space $L^2(\mathcal{M}) \oplus L^2(\mathcal{TM})$ to itself, where we consider the domain of definition of $D_XF((\phi_c, \mathbf{0}), 0)$ to be $H^2(\mathcal{M}) \oplus H^2(\mathcal{TM})$. Indeed, the operator $D_XF((\phi_c, \mathbf{0}), 0)$ induces the bilinear form

$$B((u_1, \mathbf{v}_1), (u_2, \mathbf{v}_2)) : (H^1(\mathcal{M}) \oplus H^1(\mathcal{TM})) \times (H^1(\mathcal{M}) \oplus H^1(\mathcal{TM})) \to \mathbb{R},$$

where $\langle \cdot, \cdot \rangle$ is the inner product associated with $L^2(\mathcal{M}) \oplus L^2(\mathcal{TM})$ and

$$B((u_1, \mathbf{v}_1), (u_2, \mathbf{v}_2)) = \left\langle \begin{bmatrix} -\Delta & \tilde{\mathbb{L}} \\ \mathbf{0} & \mathbb{L} \end{bmatrix} \begin{bmatrix} u_1 \\ \mathbf{v}_1 \end{bmatrix}, \begin{bmatrix} u_2 \\ \mathbf{v}_2 \end{bmatrix} \right\rangle. \tag{4.24}$$

Paralleling the discussion in 10.2.3, we first show there exists constants $C, c > 0$ such that

$$B((u, \mathbf{v}), (u, \mathbf{v})) + c\langle (u, \mathbf{v}), (u, \mathbf{v}) \rangle \geq C \|(u, \mathbf{v})\|^2_{H^1(\mathcal{M}) \oplus H^1(\mathcal{TM})}.$$

Let $c > 0$ be a constant to be determined. Then

$$\begin{aligned} &B((u, \mathbf{v}), (u, \mathbf{v})) + c\langle (u, \mathbf{v}), (u, \mathbf{v}) \rangle \\ &= \int_{\mathcal{M}} \left(D^a u D_a u - \frac{1}{4} u \phi_c^{-7} \sigma_{ab} (\mathcal{L}v)^{ab} + (\mathcal{L}v)^{ab} (\mathcal{L}v)_{ab} + cu^2 + cv^a v_a \right) dV_g \\ &\geq \int_{\mathcal{M}} \left(D^a u D_a u - \frac{1}{16c\epsilon} u^2 - \epsilon \phi_c^{-14} (\sigma_{ab} (\mathcal{L}v)^{ab})^2 + (\mathcal{L}v)^{ab} (\mathcal{L}v)_{ab} + cu^2 + cv^a v_a \right) dV_g, \end{aligned} \tag{4.25}$$

where the above inequality follows from an application of Young's inequality. The Schwartz inequality and the definition of $\mathcal{L}$ then imply that

$$\sigma_{ab} (\mathcal{L}v)^{ab} = \langle \sigma, \mathcal{L}\mathbf{v} \rangle_g \leq C |\sigma| |D\mathbf{v}|.$$

Therefore

$$\int_{\mathcal{M}} \epsilon \phi_c^{-14} (\sigma_{ab} (\mathcal{L}v)^{ab})^2 \leq c(\epsilon) \|\mathbf{v}\|^2_{1,2}, \tag{4.26}$$

where $c(\epsilon) \to 0$ as $\epsilon \to 0$. Combining (6.4) and (6.5) we have that

$$\begin{aligned} &B((u, \mathbf{v}), (u, \mathbf{v})) + c\langle (u, \mathbf{v}), (u, \mathbf{v}) \rangle \geq \\ &(1 - c(\epsilon)) \|\mathbf{v}\|^2_{1,2} + \|Du\|^2_{0,2} + (c - \frac{1}{16\epsilon}) \|u\|^2_{0,2} \geq C(\|\mathbf{v}\|^2_{1,2} + \|u\|^2_{1,2}), \end{aligned} \tag{4.27}$$

where the final inequality holds by choosing ϵ sufficiently small and c sufficiently large.

The above discussion tells us that the bilinear form

$$B((u, \mathbf{v}), (u, \mathbf{v})) + c\langle (u, \mathbf{v}), (u, \mathbf{v}) \rangle$$

is coercive on $H^1(\mathcal{M}) \oplus H^1(\mathcal{TM})$. The Lax-Milgram theorem implies that the problem

$$(D_X F((\phi_c, \mathbf{0}), 0) + cI) \begin{bmatrix} u \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} f \\ \mathbf{g} \end{bmatrix}$$

has a unique weak solution $(u, \mathbf{v}) \in H^1(\mathcal{M}) \oplus H^1(\mathcal{TM})$ for each $(f, \mathsf{g}) \in L^2(\mathcal{M}) \oplus L^2(\mathcal{TM})$, and elliptic regularity gives us that $(u, \mathbf{v}) \in H^2(\mathcal{M}) \oplus H^2(\mathcal{TM})$. Therefore we conclude that the operator $D_X F((\phi_c, \mathbf{0}), 0) + cI$ is a bijection between $H^2(\mathcal{M}) \oplus H^2(\mathcal{TM})$ and $L^2(\mathcal{M}) \oplus L^2(\mathcal{TM})$. We are the able to conclude that

$$(D_X F((\phi_c, \mathbf{0}), 0) + cI)^{-1} \quad \text{exists and is compact.}$$

Paralleling the discussion in Appendix 10.2.3, we can then conclude that the operator $D_X F((\phi_c, \mathbf{0}), 0)$ is a Fredholm operator between $H^2(\mathcal{M}) \oplus H^2(\mathcal{TM})$ and $L^2(\mathcal{M}) \oplus L^2(\mathcal{TM})$. Using the fact that $C^{0,\alpha}(\mathcal{M}) \oplus C^{0,\alpha}(\mathcal{TM})$ embeds continuously into $L^2(\mathcal{M}) \oplus L^2(\mathcal{TM})$ and invoking classical Schauder estimates, an argument similar to the argument in 10.2.3 implies that $D_X F((\phi_c, \mathbf{0}), 0)$ is Fredholm operator between the spaces $C^{2,\alpha}(\mathcal{M}) \oplus C^{2,\alpha}(\mathcal{TM})$ and $C^{0,\alpha}(\mathcal{M}) \oplus C^{0,\alpha}(\mathcal{TM})$. By applying the same argument to $D_X F((\phi_c, \mathbf{0}), 0)^*$, we can also conclude that this operator is a Fredholm operator between $C^{2,\alpha}(\mathcal{M}) \oplus C^{2,\alpha}(\mathcal{TM})$ and $C^{0,\alpha}(\mathcal{M}) \oplus C^{0,\alpha}(\mathcal{TM})$.

If $L = D_X F((\phi_c, \mathbf{0}), 0)$, then the fact that both L, L^* are Fredholm operators from $H^2(\mathcal{M}) \oplus H^2(\mathcal{TM}) \to L^2(\mathcal{M}) \oplus L^2(\mathcal{TM})$ allows us to decompose $L^2(\mathcal{M}) \oplus L^2(\mathcal{TM})$ as in (6.1). Therefore, regarding $C^{2,\alpha}(\mathcal{M}) \oplus C^{2,\alpha}(\mathcal{TM})$ and $C^{0,\alpha}(\mathcal{M}) \oplus C^{0,\alpha}(\mathcal{TM})$ as subspaces of $L^2(\mathcal{M}) \oplus L^2(\mathcal{TM})$, we obtain the following decompositions that are orthogonal with respect to the $L^2(\mathcal{M}) \oplus L^2(\mathcal{TM})$- inner product:

$$\begin{aligned} C^{2,\alpha}(\mathcal{M}) \oplus C^{2,\alpha}(\mathcal{TM}) &= \ker(L) \oplus (R(L^*) \cap (C^{2,\alpha}(\mathcal{M}) \oplus C^{2,\alpha}(\mathcal{TM}))), \\ C^{0,\alpha}(\mathcal{M}) \oplus C^{0,\alpha}(\mathcal{TM}) &= \ker(L^*) \oplus (R(L) \cap (C^{0,\alpha}(\mathcal{M}) \oplus C^{0,\alpha}(\mathcal{TM}))). \end{aligned} \tag{4.28}$$

In the above decomposition, $L = D_X F((\phi_c, \mathbf{0}), 0)$ and $\ker(L), R(L), \ker(L^*)$ and $R(L^*)$ are all regarded as subspaces of $L^2(\mathcal{M}) \oplus L^2(\mathcal{TM})$.

5. Proofs of the Main Results

5.1. Proof of Theorem 3.3: Bifurcation and non-uniqueness in the CMC case. We are now ready to prove Theorem 3.3. In the CMC case, our system (3.1) with $\rho = \rho_c$ reduces to

$$G(\phi, \lambda) = -\Delta\phi + a_R\phi + \lambda^2 a_\tau \phi^5 - \frac{1}{8}\sigma^2\phi^{-7} - 2\pi\rho_c e^{-\lambda}\phi^5. \tag{5.1}$$

To prove that solutions to (7.1) are non-unique, we will apply the Liapunov-Schmidt reduction outlined in Section 2.4 and then invoke Theorem 2.5 and Proposition 2.7.

By Theorem 3.1 and Remark 5.2, we know that $D_\phi G(\phi_c, 0) = -\Delta$. It follows that $\dim \ker(D_\phi G(\phi_c, 0)) = \dim \ker(D_\phi G(\phi_c, 0)^*) = 1$, where both spaces are spanned by $\phi = 1$.

Using the notation from Section 2.4, we can apply the Liapunov-Schmidt Reduction, where $\hat{v}_0 = 1$ is a basis of $\ker(D_\phi G(\phi_c, 0)) = \ker(D_\phi G(\phi_c, 0)^*)$. By the discussion in Section 6 and appendix 10.2.3, we can decompose $X = C^{2,\alpha}(\mathcal{M}) = X_1 \oplus X_2$ and $Y = C^{0,\alpha}(\mathcal{M}) = Y_1 \oplus Y_2$, where

$$\begin{aligned} &X_1 = \ker(D_\phi G(\phi_c, 0)), \quad X_2 = R(D_\phi G(\phi_c, 0)^*) \cap C^{2,\alpha}(\mathcal{M}), \\ &Y_1 = R(D_\phi G(\phi_c, 0)) \cap C^{0,\alpha}(\mathcal{M}), \quad \text{and} \quad Y_2 = \ker(D_\phi G(\phi_c, 0)^*). \end{aligned} \tag{5.2}$$

Letting $P : X \to X_1$ and $Q : Y \to Y_2$ be projection operators as in Section 2.4, and writing $\phi = P\phi + (I - P)\phi = v + w$, the Implicit Function Theorem applied to

$$(I - Q)G(v + w, \lambda) = 0, \tag{5.3}$$

implies that $w = \psi(v, \lambda)$ in a neighborhood of $(\phi_c, 0)$ and $0 = \psi(\phi_c, 0)$. Plugging $\psi(v, \lambda)$ into

$$QG(v + w, \lambda) = 0,$$

we obtain

$$\Phi(v, \lambda) = QG(v + \psi(v, \lambda), \lambda) = 0. \tag{5.4}$$

All solutions to $G(\phi, \lambda) = 0$ in a neighborhood of $(\phi_c, 0)$ must satisfy Eq. (7.4).

We now observe that $D_\lambda G(\phi_c, 0) = 2\pi\rho_c\phi_c^5 \neq 0$. This implies that

$$D_\lambda \Phi(\phi_c, 0) = QD_\lambda G(\phi_c, 0) = 2\pi\rho_c\phi_c^5 \neq 0, \tag{5.5}$$

given that Q is the projection onto Y_2 and Y_2 is spanned by the constant function 1. The Implicit Function Theorem applied to Eq. (7.4) implies that there exists a function $\gamma : U_1 \to V_1$ such that $U_1 \subset X_1$, $V_1 \subset \mathbb{R}$ and $\gamma(v) = \lambda$ in a neighborhood of ϕ_c with $\gamma(\phi_c) = 0$.

Therefore (7.4) becomes

$$g(v) = QG(v + \psi(v, \gamma(v)), \gamma(v)), \tag{5.6}$$

and by writing $v = s + \phi_c$, which we can do for $s \in (-\delta, \delta)$ with $\delta > 0$ sufficiently small, we obtain

$$g(s) = QG(s + \phi_c + \psi(s + \phi_c, \gamma(s + \phi_c)), \gamma(s + \phi_c)) = 0. \tag{5.7}$$

This implies that solutions to $G(\phi, \lambda) = 0$ are given by $g(s) = 0$ in a neighborhood of $(\phi_c, 0)$, where

$$\begin{aligned} \phi(s) &= s + \phi_c + \psi(s + \phi_c, \gamma(s + \phi_c)), \\ \lambda(s) &= \gamma(s + \phi_c) \end{aligned} \tag{5.8}$$

determine a differentiable solution curve through $(\phi_c, 0)$.

Equation (7.8) gives us a fairly explicit representation of the continuously differentiable curve $\{\phi(s), \lambda(s)\}$ provided by Theorem 2.5. However, by applying Proposition 2.7 we can determine that $\ddot{\lambda}(0) \neq 0$ to obtain even more information about $\{\phi(s), \lambda(s)\}$. We observe that

$$D^2_{\phi\phi} G(\phi_c, 0)[\hat{v}_0, \hat{v}_0] = -7\sigma^2 \phi_c^{-9} - 40\pi\rho_c \phi_c^3 \neq 0. \tag{5.9}$$

Therefore

$$-7\sigma^2\phi_c^{-9} - 40\pi\rho_c\phi_c^3 \in Y_2 \Longrightarrow D^2_{\phi\phi}G(\phi_c, 0)[\hat{v}_0, \hat{v}_0] \notin R(D_\phi G(\phi_c, 0)) = Y_1,$$

given that $Y_1 \perp Y_2$. Proposition 2.7 implies that $\ddot{\lambda}(0) \neq 0$ and that a saddle node bifurcation occurs at $(\phi_c, 0)$.

We now combine (7.8) and the fact that $\ddot{\lambda}(0) \neq 0$ to obtain a more explicit representation to the solution curve $\{\phi(s), \lambda(s)\}$ in a neighborhood of $(\phi_c, 0)$. Define the function

$$f(s) = \psi(s + \phi_c, \gamma(s + \phi_c)). \tag{5.10}$$

Then by Propositions 2.6 and 2.7 we have that

$$\begin{aligned} &f(0) = 0, \quad \text{and} \quad \lambda(0) = \gamma(\phi_c) = 0, \\ &\dot{\lambda}(0) = \frac{d}{ds}\lambda(s)\bigg|_{s=0} = D_v\gamma(\phi_c) = 0, \\ &\dot{f}(0) = \frac{d}{ds}f(s)\bigg|_{s=0} = D_v\psi(\phi_c, 0) + D_\lambda\psi(\phi_c, 0)D_v\gamma(\phi_c) = 0. \end{aligned} \tag{5.11}$$

Therefore the function $f(s) = O(s^2)$. By computing a Taylor expansion of $\lambda(s)$ about $s = 0$ and using Eq. (7.11) and Eq. (7.8), we find that for $s \in (-\delta, \delta)$,

$$\begin{aligned} \phi(s) &= \phi_c + s + O(s^2), \\ \lambda(s) &= \frac{1}{2}\ddot{\lambda}(0)s^2 + O(s^3), \end{aligned} \tag{5.12}$$

where $\ddot{\lambda}(0) \neq 0$.

Based on the form of $\phi(s)$ and $\lambda(s)$ in Eq. (7.12), there exists a $\delta' \in (0, \delta)$ such that $\phi(s) < 0$, $\lambda(s) > 0$ for all $s \in [-\delta', 0)$, and $\phi(s) > 0$, $\lambda(s) > 0$ for all $s \in (0, \delta']$. Letting $M = \min\{M_1, M_2\}$, where

$$M_1 = \sup_{s\in[-\delta',0]} \lambda(s) \quad \text{and} \quad M_2 = \sup_{s\in[0,\delta']} \lambda(s),$$

the Intermediate Value Theorem then implies that for all $\lambda_0 \in (0, M)$, there exists $s_1, s_2 \in [-\delta', \delta']$, $s_1 \neq s_2$, such that $\lambda(s_1) = \lambda(s_2) = \lambda_0$. Based on how we chose δ', we also have that $\phi(s_1) \neq \phi(s_2)$. This completes the proof of Theorem 3.3.

5.2. **Proof of Theorem 3.4: Bifurcation and non-uniqueness in the non-CMC case.** In this section we will show that solutions to $F((\phi, \mathbf{w}), 0) = 0$ for the full system

$$F((\phi,\mathbf{w}),\lambda) = \left[\begin{array}{c} -\Delta\phi + a_R\phi + \lambda^2 a_\tau \phi^5 - a_{\mathbf{w}}\phi^{-7} - 2\pi\rho e^{-\lambda}\phi^5 \\ \mathbb{L}\mathbf{w} + \lambda b_\tau^a \phi^6 \end{array}\right] \tag{5.13}$$

are non-unique, where $\tau \in C^{1,\alpha}(\mathcal{M})$ is a non-constant function. Our approach is similar to that of the CMC case: we apply a Liapunov-Schmidt reduction to Eq. (8.1) to determine an explicit solution curve through the point $((\phi_c, \mathbf{0}), 0)$. The form of this curve will imply that solutions to the system (8.1) are non-unique.

By Proposition 5.1 we know that $\ker D_X F((\phi_c, \mathbf{0}), 0)$ takes the form

$$D_X F(\phi_c, \mathbf{0}, 0) = \left[\begin{array}{cc} -\Delta & \tilde{\mathbb{L}} \\ 0 & \mathbb{L} \end{array}\right],$$

where $\tilde{\mathbb{L}}h = -\frac{1}{4}\phi_c^{-7}\sigma_{ab}(\mathcal{L}h)^{ab}$. Corollary 5.4 gives us that $\ker(D_X F((\phi_c, \mathbf{0}), 0))$ and $\ker(D_X F((\phi_c, \mathbf{0}), 0)^*)$ are spanned by $\hat{v}_0 = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$.

Using the notation from Section 2.4, we apply the Liapunov-Schmidt Reduction. By the decomposition (6.7), we have that

$$X = C^{2,\alpha}(\mathcal{M}) \oplus C^{2,\alpha}(\mathcal{TM}) = X_1 \oplus X_2,$$

and

$$Y = C^{0,\alpha}(\mathcal{M}) \oplus C^{0,\alpha}(\mathcal{TM}) = Y_1 \oplus Y_2,$$

where

$$X_1 = \ker(D_X F((\phi_c, \mathbf{0}), 0)), \tag{5.14}$$

$$X_2 = R(D_X F((\phi_c, \mathbf{0}), 0)^*) \cap (C^{2,\alpha}(\mathcal{M}) \oplus C^{2,\alpha}(\mathcal{TM})), \tag{5.15}$$

$$Y_1 = R(D_X F((\phi_c, \mathbf{0}), 0)) \cap (C^{0,\alpha}(\mathcal{M}) \oplus C^{0,\alpha}(\mathcal{TM})), \tag{5.16}$$

$$Y_2 = \ker(D_X F((\phi_c, \mathbf{0}), 0)^*). \tag{5.17}$$

Let $P : X \to X_1$ and $Q : Y \to Y_2$ be the projection operators defined using $\hat{v}_0$ as in Section 2.4. Then by writing

$$\left[\begin{array}{c} \phi \\ \mathbf{w} \end{array}\right] = P\left[\begin{array}{c} \phi \\ \mathbf{w} \end{array}\right] + (I-P)\left[\begin{array}{c} \phi \\ \mathbf{w} \end{array}\right] = v + y,$$

the Implicit Function Theorem applied to

$$(I - Q)F(v + y, \lambda) = 0, \tag{5.18}$$

implies that solutions to $F((\phi, \mathbf{w}), \lambda) = 0$ satisfy

$$\Phi(v, \lambda) = QF(v + \psi(v, \lambda), \lambda) = 0 \tag{5.19}$$

in a neighborhood of $((\phi_c, \mathbf{0}), 0)$, where $y = \psi(v, \lambda)$ in this neighborhood and $(0, \mathbf{0}) = \psi((\phi_c, \mathbf{0}), 0)$.

We now observe that

$$D_\lambda F((\phi_c, \mathbf{0}), 0) = \left[\begin{array}{c} 2\pi\rho_c\phi_c^5 \\ b_\tau^a\phi_c^6 \end{array}\right] \notin Y_1,$$

due to the fact that

$$\begin{bmatrix} 2\pi\rho_c\phi_c^5 \\ 0 \end{bmatrix} \in Y_2 \quad \text{and} \quad Y_1 \perp Y_2.$$

This implies that

$$D_\lambda\Phi((\phi_c, \mathbf{0}), 0) = QD_\lambda F((\phi_c, \mathbf{0}), 0) = \begin{bmatrix} 2\pi\rho_c\phi_c^5 \\ 0 \end{bmatrix} \neq 0, \tag{5.20}$$

given that Q is the projection onto Y_2. The Implicit Function Theorem again implies that there exists a function $\gamma : U_1 \to V_1$, where $(\phi_c, \mathbf{0}) \in U_1 \subset X_1$, $V_1 \subset \mathbb{R}$ and $\gamma(v) = \lambda$ in U_1 with $\gamma(\phi_c, \mathbf{0}) = 0$. Using this fact, Eq. (8.7) becomes

$$g(v) = QF(v + \psi(v, \gamma(v)), \gamma(v)) = 0, \tag{5.21}$$

and by writing

$$v = (s + \phi_c)\hat{v}_0 = s\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \phi_c \\ 0 \end{bmatrix},$$

for $s \in (-\delta, \delta)$ with $\delta > 0$ sufficiently small, we then obtain

$$g(s) = QF\left(s\hat{v}_0 + \phi_c\hat{v}_0 + \psi(s\hat{v}_0 + \phi_c\hat{v}_0, \gamma(s\hat{v}_0 + \phi_c\hat{v}_0)), \gamma(s\hat{v}_0 + \phi_c\hat{v}_0)\right) = 0. \tag{5.22}$$

This implies that solutions to $F((\phi, \mathbf{0}), \lambda) = 0$ in a neighborhood of $((\phi_c, \mathbf{0}), 0)$ satisfy $g(s) = 0$, where

$$\begin{bmatrix} \phi(s) \\ \mathbf{w}(s) \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \phi_c \\ 0 \end{bmatrix} + \psi\left(s\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \phi_c \\ 0 \end{bmatrix}, \gamma\left(s\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \phi_c \\ 0 \end{bmatrix}\right)\right), \tag{5.23}$$

$$\lambda(s) = \gamma\left(s\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \phi_c \\ 0 \end{bmatrix}\right),$$

determine a smooth solution curve through $((\phi_c, \mathbf{0}), 0)$.

As in the CMC case, we seek additional information so that we can further analyze the solution curve (8.11). Now we apply Proposition 2.7 to determine information about $\ddot{\lambda}(0)$, and then we will expand the function

$$f(s) = \psi((s + \phi_c)\hat{v}_0, \gamma((s + \phi_c)\hat{v}_0)) \tag{5.24}$$

as a Taylor series to obtain a more explicit representation of $\{(\phi(s), \mathbf{w}(s)), \lambda(s)\}$.

Taking the second derivative of $F((\phi, \mathbf{w}), \lambda)$, we have that

$$D^2_{XX}F((\phi_c, \mathbf{0}), 0)[\hat{v}_0, \hat{v}_0] = \begin{bmatrix} -7\sigma^2\phi_c^{-9} - 40\pi\rho_c\phi_c^3 \\ \mathbf{0} \end{bmatrix} \in Y_2. \tag{5.25}$$

Given that the vector (8.13) lies in Y_2 and $Y_1 \perp Y_2$,

$$D^2_{XX}F((\phi_c, \mathbf{0}), 0)[\hat{v}_0, \hat{v}_0] \notin Y_1.$$

We can therefore apply Proposition 2.7 to conclude that $\ddot{\lambda}(0) \neq 0$.

Our next goal is to expand the function $f(s)$ as a Taylor series about 0. In order to do this, we use (8.11), Proposition 2.6 and the fact that $\ddot{\lambda}(0) \neq 0$ to obtain information about coefficients in this expansion. In particular, the objective is to determine information about the coefficient of the second order term in the expansion of $f(s)$.

By differentiating

$$(I - Q)F(v + \psi(v, \lambda), \lambda) = 0,$$

with respect to λ and evaluating the resulting expression at $((\phi_c, \mathbf{0}), 0)$, we obtain

$$(I - Q)D_XF((\phi_c, \mathbf{0}), 0)D_\lambda\psi((\phi_c, \mathbf{0}), 0) + (I - Q)D_\lambda F((\phi_c, \mathbf{0}), 0) = 0. \tag{5.26}$$

Given that

$$D_\lambda F((\phi_c, \mathbf{0}), 0) = \begin{bmatrix} 2\pi\rho_c\phi_c^5 \\ b_\tau^a \phi_c^6 \end{bmatrix},$$

and Q is the projection operator onto Y_2, which is spanned by $\begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix}$, we have that

$$(I - Q)D_\lambda F((\phi_c, \mathbf{0}), 0) = \begin{bmatrix} 0 \\ b_\tau^a \phi_c^6 \end{bmatrix}. \tag{5.27}$$

Equations (8.15) and (8.14) imply that

$$(I - Q)D_X F((\phi_c, \mathbf{0}), 0)D_\lambda \psi((\phi_c, \mathbf{0}), 0) = -\begin{bmatrix} 0 \\ b_\tau^a \phi_c^6 \end{bmatrix}. \tag{5.28}$$

Given that $D_X F((\phi_c, \mathbf{0}), 0)$ has the form (5.4) and the operator $\mathbb{L}$ is invertible, Eq. (8.16) implies that

$$D_\lambda \psi((\phi_c, \mathbf{0}), 0) = \begin{bmatrix} u(x) \\ \mathbf{v}(x) \end{bmatrix}, \quad \text{with} \quad \mathbf{v}(x) \neq \mathbf{0}. \tag{5.29}$$

As we shall see, this fact implies that $\mathbf{w}(s)$ has quadratic terms in s.

We have one last piece of data left to determine the coefficient of the second order term in the Taylor expansion of $f(s)$. Differentiating $(I - Q)F(v + \psi(v, \lambda), \lambda) = 0$ twice with respect to v, evaluating at $((\phi_c, \mathbf{0}), 0)$ and applying the resulting bilinear form to $\hat{v}_0$, we obtain

$$\begin{aligned} &(I - Q)D^2_{XX} F((\phi_c, \mathbf{0}), 0)[\hat{v}_0, \hat{v}_0] + \\ &(I - Q)D_X F((\phi_c, \mathbf{0}), 0)D^2_{vv}\psi((\phi_c, \mathbf{0}), 0)[\hat{v}_0, \hat{v}_0] = 0. \end{aligned} \tag{5.30}$$

By Eq. (8.13) we know that $D^2_{XX} F((\phi_c, \mathbf{0}), 0)[\hat{v}_0, \hat{v}_0] \in Y_2$. Because $(I - Q)$ projects onto Y_1 and $Y_1 \perp Y_2$, we have that

$$(I - Q)D^2_{XX} F((\phi_c, \mathbf{0}), 0)[\hat{v}_0, \hat{v}_0] = 0. \tag{5.31}$$

Equations (8.19) and (8.18) and the invertibility of $(I-Q)D_X F((\phi_c, \mathbf{0}), 0)$ as an operator from X_2 to Y_1 imply that

$$D^2_{vv}\psi((\phi_c, \mathbf{0}), 0)[\hat{v}_0, \hat{v}_0] = 0. \tag{5.32}$$

This was the final piece of information that we needed to to determine the second order expansion of $f(s)$.

We now expand the function $f(s)$ in Eq. (8.12) about $s = 0$. We have that

$$f(0) = \psi((\phi_c, \mathbf{0}), 0) = \begin{bmatrix} 0 \\ \mathbf{0} \end{bmatrix}, \tag{5.33}$$

$$\dot{f}(0) = D_v\psi((\phi_c, \mathbf{0}), 0)\hat{v}_0 + D_\lambda\psi((\phi_c, \mathbf{0}), 0))D_v\gamma(\phi_c, \mathbf{0})\hat{v}_0 = \begin{bmatrix} 0 \\ \mathbf{0} \end{bmatrix},$$

$$\begin{aligned} \ddot{f}(0) &= D^2_{vv}\psi((\phi_c, \mathbf{0}), 0)[\hat{v}_0, \hat{v}_0] + D^2_{v\lambda}\psi((\phi_c, \mathbf{0}), 0)[\hat{v}_0, D_v\gamma(\phi_c, \mathbf{0})\hat{v}_0] \\ &+ D^2_{\lambda v}\psi((\phi_c, \mathbf{0}), 0)[D_v\gamma(\phi_c, \mathbf{0})\hat{v}_0, \hat{v}_0] + D_\lambda\psi((\phi_c, \mathbf{0}), 0)D^2_{vv}\gamma(\phi_c, \mathbf{0})[\hat{v}_0, \hat{v}_0] \\ &+ D^2_{\lambda\lambda}\psi((\phi_c, \mathbf{0}), 0)[D_v\gamma(\phi_c, \mathbf{0})\hat{v}_0, D_v\gamma(\phi_c, \mathbf{0})\hat{v}_0] \\ &= D_\lambda\psi((\phi_c, \mathbf{0}), 0)D^2_{vv}\gamma(\phi_c, \mathbf{0})[\hat{v}_0, \hat{v}_0] = D_\lambda\psi((\phi_c, \mathbf{0}), 0)\ddot{\lambda}(0) \neq \begin{bmatrix} 0 \\ \mathbf{0} \end{bmatrix}, \end{aligned}$$

where $\ddot{f}(0)$ simplifies as a result of Proposition 2.7, Eq. (8.17) and Eq. (8.20), which imply

$$D_v\psi((\phi_c,\mathbf{0}),0) = 0, \qquad D_v\gamma(\phi_c,\mathbf{0}) = 0, \tag{5.34}$$
$$D^2_{vv}\psi((\phi_c,\mathbf{0}),0)[\hat{v}_0,\hat{v}_0] = \begin{bmatrix} 0 \\ \mathbf{0} \end{bmatrix}, \qquad D_\lambda\psi((\phi_c,\mathbf{0}),0) \neq \begin{bmatrix} 0 \\ \mathbf{0} \end{bmatrix}.$$

Therefore it follows that

$$f(s) = \frac{1}{2}(D_\lambda\psi(\phi_c\hat{v}_0,\gamma(\phi_c\hat{v}_0))\ddot{\lambda}(0))s^2 + O(s^3) = \begin{bmatrix} \frac{1}{2}u(x)\ddot{\lambda}(0) \\ \frac{1}{2}\mathbf{v}(x)\ddot{\lambda}(0) \end{bmatrix} s^2 + O(s^3), \tag{5.35}$$

where we identify $D_\lambda\psi((\phi_c,\mathbf{0}),0)$ with the vector $\begin{bmatrix} u(x) \\ \mathbf{v}(x) \end{bmatrix}$ in $C^{2,\alpha}(\mathcal{M}) \oplus C^{2,\alpha}(\mathcal{TM})$. By Eq. (8.17) we have that $\mathbf{v}(x) \neq 0$ and expanding out $\lambda(s)$ as a second order Taylor series about $s = 0$ we obtain

$$\lambda(s) = \frac{1}{2}\ddot{\lambda}(0)s^2 + O(s^3). \tag{5.36}$$

Putting together (8.11), (8.23) and (8.24) we find that solutions to $F((\phi,\mathbf{w}),\lambda) = 0$ in a neighborhood of $((\phi_c,\mathbf{0}),0)$ take the form

$$\phi(s) = \phi_c + s + \frac{1}{2}\ddot{\lambda}(0)u(x)s^2 + O(s^3), \tag{5.37}$$
$$\mathbf{w}(s) = \frac{1}{2}\ddot{\lambda}(0)\mathbf{v}(x)s^2 + O(s^3), \tag{5.38}$$
$$\lambda(s) = \frac{1}{2}\ddot{\lambda}(0)s^2 + O(s^3), \tag{5.39}$$

where $s \in (-\delta,\delta)$ for sufficiently small $\delta > 0$.

By analyzing the solution curve (8.25)-(8.27) as we did for the curve (7.12) in the proof of Theorem 3.3, we can conclude that solutions to the system (8.1) are non-unique. This completes the proof of Theorem 3.4.

6. Summary

We began in Section 2 by introducing our notation for function spaces and presenting the basic concepts from functional analysis and bifurcation theory that we used throughout this paper. In particular, we gave an outline of the Liapunov-Schmidt reduction that was the basis of our non-uniqueness arguments. Then in Section 3 we presented our main results, which consisted of the existence of a critical solution where the linearizations of our system

$$F((\phi,\mathbf{w}),\lambda) = \begin{bmatrix} -\Delta\phi + a_R\phi + \lambda^2 a_\tau\phi^5 - a_{\mathbf{w}}\phi^{-7} - 2\pi\rho e^{-\lambda}\phi^5 \\ \mathbb{L}\mathbf{w} + \lambda b^a_\tau\phi^6 \end{bmatrix}, \tag{6.1}$$

developed a one-dimensional kernel and non-uniqueness results for solutions to $F((\phi,\mathbf{w}),0) = 0$ in both the CMC and non-CMC cases. We then set about proving these results in the following sections. In Section 4 we showed that in the CMC case there exists a critical density ρ_c for the operator

$$G(\phi,\lambda) = -\Delta\phi + a_R\phi + \lambda^2 a_\tau - a_{\mathbf{w}}\phi^{-7} - 2\pi\rho e^{-\lambda}\phi^5. \tag{6.2}$$

This density satisfied the property that if $|\lambda|$ was sufficiently small, then $\rho > \rho_c$ and $\lambda < 0$ implied that there was no solution to $G(\phi,\lambda) = 0$, and if $\rho \le \rho_c$ and $\lambda \ge 0$ then there was a solution. This result provided the foundation in Section 5 for showing that the linearization of (9.1) developed a one-dimensional kernel. Then in Section 6 we briefly discussed the Fredholm properties of the linearized operators $D_X F((\phi_c, \mathbf{0}), 0)$ and $D_\phi G(\phi_c, 0)$ on the Banach spaces on which they are defined.

In Section 7 we proved the first of our non-uniqueness results. We showed that in the event that the mean curvature was constant, the decoupled system (9.2) exhibited non-uniqueness. This was indicated by the fact that the solution curve through the point $(\phi_c, 0)$ had the form

$$\begin{aligned} \phi(s) &= \phi_c + s + O(s^2), \\ \lambda(s) &= \frac{1}{2}\ddot{\lambda}(0)s^2 + O(s^3), \end{aligned} \tag{6.3}$$

which implied that a saddle-node bifurcation occurred at the point $(\phi_c, 0)$. We were able to determine the explicit form of the solution curve (9.3) by applying a Liapunov-Schmidt reduction to (9.2) at the point $(\phi_c, 0)$, which was possible given that the operator $D_\phi G(\phi_c, 0)$ had a one-dimensional kernel. Similarly, in Section 8 we showed that when the mean curvature τ was an arbitrary, continuously differentiable function, solutions to $F((\phi, \mathbf{w}), \lambda) = 0$ were non-unique. Again, this followed because we explicitly computed the solution curve through the point $((\phi_c, \mathbf{0}), 0)$. In Section 8 we found that the solution curve through $((\phi_c, \mathbf{0}), 0)$ had the form

$$\phi(s) = \phi_c + s + \frac{1}{2}\ddot{\lambda}(0)u(x)s^2 + O(s^3), \tag{6.4}$$

$$\mathbf{w}(s) = \frac{1}{2}\ddot{\lambda}(0)\mathbf{v}(x)s^2 + O(s^3), \tag{6.5}$$

$$\lambda(s) = \frac{1}{2}\ddot{\lambda}(0)s^2 + O(s^3), \tag{6.6}$$

which we demonstrated by applying a Liapunov-Schmidt reduction to the system (9.1) at the point $((\phi_c, \mathbf{0}), 0)$. Again, this was possible because of our work in Section 4 where we showed that the linearization $D_X F((\phi_c, \mathbf{0}), 0)$ had a one-dimensional kernel.

The importance of these non-uniqueness results is that they demonstrate first and foremost that the conformal formulation with unscaled source terms is undesirable given that solutions for this formulation will not allow us to uniquely parametrize physical solutions to the Einstein constraint equations. Additionally, this paper helps build on the work of Walsh in [15] by expanding the understanding of how bifurcation techniques can be applied to the various conformal formulations of the constraint equations. This work is also interesting in that the analysis conducted here helps clarify the ideas of O'Murchadha et. al. in [4] by showing how terms with "the wrong sign" that contribute to the non-monotonicity (non-convexity of the corresponding energy) of the nonlinearity in the Hamiltonian constraint directly contribute to the non-uniqueness of solutions. Finally, it is hope of the authors that this work will also help to lay the foundation for future analysis of the uniqueness properties of the Conformal Thin Sandwich method and the far-from-CMC solution framework established in [7, 8].

7. Appendix

7.1. **Banach Calculus and the Implicit Function Theorem.** Here we give a brief review of some basic tools from functional analysis. The following results are presented without proof and are taken from [16]. We begin with some notation.

Suppose that X and Y are Banach spaces and $U \subset X$ is a neighborhood of 0. For a given map $f : U \subset X \to Y$, we say that

$$f(x) = o(\|x\|), \quad x \to 0 \quad \text{iff} \quad r(x)/\|x\| \to 0 \ \text{ as } \ x \to 0.$$

We write $L(X,Y)$ for the class of continuous linear maps between the Banach spaces X and Y.

Definition 7.1. *Let $U \subset X$ be a neighborhood of x and suppose that X and Y are Banach spaces.*

(1) *We say that a map $f : U \to Y$ is* **F-differentiable** *or* **Fréchet differentiable** *at x iff there exists a map $T \in L(X,Y)$ such that*
$$f(x+h) - f(x) = Th + o(\|h\|), \quad \text{as } h \to 0,$$
for all h in some neighborhood of zero. If it exists, T is called the **F-derivative** *or* **Fréchet derivative** *of f and we define $f'(x) = T$. If f is Fréchet differentiable for all $x \in U$ we say that f is Fréchet differentiable in U. Finally, we define the* **F-differential** *at x to be $df(x;h) = f'(x)h$.*

(2) *The map f is* **G-differentiable** *or* **Gâteaux differentiable** *at x iff there exists a map $T \in L(X,Y)$ such that*
$$f(x+tk) - f(x) = tTk + o(t), \quad \text{as } t \to 0,$$
for all k with $\|k\| = 1$ and all real numbers t in some neighborhood of zero. If it exists, T is called the **G-derivative** *or* **Gâteaux derivative** *of f and we define $f'(x) = T$. If f is G-differential for all $x \in U$ we say that f is Gâteaux differentiable in U. The* **G-differential** *at x is defined to be $d_G f(x;h) = f'(x)h$.*

Remark 7.2. *Clearly if an operator is F-differentiable, then it must also be G-differentiable. Moreover, if the G-derivative f' exists in some neighborhood of x and f' is continuous at x, then $f'(x)$ is also the F-derivative. This fact is quite useful for computing F-derivatives given that G-derivatives are easier to compute. See* [16] *for a complete discussion.*

We view F-derivatives and G-derivatives as linear maps $f'(x) : U \to L(X,Y)$. More generally, we may consider higher order derivatives maps of f. For example, the map $f''(x) : U \to L(X, L(X,Y))$ is a bilinear form. We now state some basic properties of F-derivatives. All of the following properties also hold for G-derivatives.

The Fréchet derivative satisfies many of the usual properties that we are accustomed to by doing calculus in $\mathbb{R}^n$. For example, we have the chain rule.

Proposition 7.3 (Chain Rule)**.** *Suppose that X, Y and Z are Banach spaces and assume that $f : U \subset X \to Y$ and $g : V \subset Y \to Z$ are differentiable on U and V resp. and that $f(U) \subset V$. Then the function $H(x) = g \circ f$, i.e. $H(x) = g(f(x))$, is differentiable where*
$$H'(x) = g'(f(x))f'(x)$$
where we write $g'(f(x))f'(x)$ for $g'(f(x)) \circ f'(x)$.

Given an operator $f : X \times Y \to Z$, we can also consider the partial derivative of f with respect to either x or y. If we fix the variable y and define $g(x) = f(x,y) : X \to Z$ and $g(x)$ is Fréchet differentiable at x, then the **partial derivative** of f with respect to x at (x,y) is $f_x(x,y) = g'(x)$. We can a make a similar definition for $f_y(x,y)$. Finally, we observe that we can express the F-differential of $f'(x,y)$ in terms of the partials by using the following formula:
$$f'(x,y)(h,k) = f_x(x,y)h + f_y(x,y)k. \tag{7.1}$$
We have the following relationship between the partial derivatives and the Fréchet derivative.

Proposition 7.4. *Suppose that $f : X \times Y \to Z$ is F-differentiable at (x, y). Then the partial F-derivatives f_x and f_y exist at (x, y) and they satisfy* (10.1)*. Moreover, if f_x and f_y both exist and are continuous in a neighborhood of (x, y) then $f'(x, y)$ exists as an F-derivative and* (10.1) *holds.*

7.1.1. *Implicit Function Theorem.* Suppose that $F : U \times V \to Z$ is a mapping with $U \subset X, V \subset Y$ and X, Y, Z are real Banach spaces. The **Implicit Function Theorem** is an extremely important tool in analyzing the nonlinear problem

$$F(x, y) = 0. \tag{7.2}$$

We present the statement of the Theorem here, the form of which is taken from [11]. For a proof see [16, **?**].

Theorem 7.5. *Let* (10.2) *have a solution $(x_0, y_0) \in U \times V$ such that the Fréchet derivative of F with respect to x at (x_0, y_0) is bijective:*

$$\begin{aligned} &F(x_0, y_0) = 0, \\ &D_x F(x_0, y_0) :\to Z \quad \textit{is bounded (continuous)} \\ &\textit{with bounded inverse.} \end{aligned} \tag{7.3}$$

Assume also that F and $D_x F$ are continuous:

$$\begin{aligned} &F \in C(U \times V, Z), \\ &D_x F \in C(U \times V, L(X, Z)), \quad \textit{where } L(X, Z) \\ &\textit{denotes the Banach space of bounded linear operators} \\ &\textit{from } X \textit{ into } Z \textit{ endowed with the operator norm.} \end{aligned} \tag{7.4}$$

Then there is a neighborhood $U_1 \times V_1 \subset U \times V$ of (x_0, y_0) and a map $f : V_1 \to U_1 \subset X$ such that

$$\begin{aligned} &f(y_0) = x_0, \\ &F(f(y), y) = 0 \quad \textit{for all } y \in V_1. \end{aligned} \tag{7.5}$$

Furthermore, $f \in C(V_1, X)$ and every solution to (10.2) *in $U_1 \times V_1$ is of the form $(f(y), y)$. Finally, if F is k-times differentiable, then f is k-times differentiable.*

7.2. **Elliptic PDE tools.** Here we assemble some useful tools for working with nonlinear elliptic partial differential equations. Throughout this section we will assume that $\mathcal{M}$ is a closed manifold with a smooth SPD metric g_{ab} and that Δ is the associated Laplace-Beltrami operator.

7.2.1. *Maximum Principle.* In this section we present a version of the maximum principle on closed manifolds. The following result is well-known, but we present it here for completeness.

Theorem 7.6. *Let $u \in C^2(\mathcal{M})$. Then if*

$$\Delta u \geq 0 \quad \textit{or} \quad \Delta u = 0 \quad \textit{or} \quad \Delta u \leq 0, \tag{7.6}$$

then u must be a constant. In particular, the problem

$$\Delta u = f(x, u),$$

has no solution if $f(x, u) \geq 0$ or $f(x, u) \leq 0$ unless $f(x, u) \equiv 0$.

Proof. See [**?**] for a proof. □

7.2.2. *Method of Sub- and Super-Solutions.* Here we present a theorem that provides a method to solve an elliptic problem of the form

$$Lu = f(x,u), \tag{7.7}$$

where

$$Lu = -\Delta u + c(x)u, \quad c(x) \in C(\mathcal{M} \times \mathbb{R})\,, \quad c(x) > 0 \tag{7.8}$$

and the function $f(x,y)$ is nonlinear in the variable y.

Theorem 7.7. *Suppose that $f : \mathcal{M} \times \mathbb{R}^+ \to \mathbb{R}$ is in $C^k(\mathcal{M} \times \mathbb{R}^+)$. Let L be of the form (10.8) and suppose that there exist functions $u_- : \mathcal{M} \to \mathbb{R}$ and $u_+ : \mathcal{M} \to \mathbb{R}$ such that the following hold:*

(1) $u_-, u_+ \in C^k(\mathcal{M})$,
(2) $0 < u_-(x) \le u_+(x) \quad \forall x \in \mathcal{M}$,
(3) $Lu_- \le f(x,u_-)$,
(4) $Lu_+ \ge f(x,u_+)$.

Then there exists a solution u to

$$Lu = f(x,u) \quad \text{on } \mathcal{M}, \tag{7.9}$$

such that

(i) $u \in C^k(\mathcal{M})$,
(ii) $u_-(x) \le u(x) \le u_+(x)$.

Proof. See [9] for a proof. □

7.2.3. *Fredholm Properties and Liapunov-Schmidt Decompositions for Elliptic Operators.* In this appendix we discuss the Fredholm properties of linear elliptic operators on a closed manifold. We use these properties to form Liapunov-Schmidt decompositions for a given elliptic operator L between certain Banach spaces. The following treatment is taken from [11].

Let $u \in C^{2,\alpha}(\mathcal{M})$ and define the elliptic operator $L : C^{2,\alpha}(\mathcal{M}) \to C^{0,\alpha}(\mathcal{M})$ by

$$Lu = -\sum_{i,j=1}^{n} (a_{ij}(x)u_{x_i})_{x_j} + \sum_{i=1}^{n} b_i(x)u_{x_i} + c(x)u, \tag{7.10}$$

where a_{ij}, b_i and c are smooth, bounded coefficients where $a_{ij} = a_{ji}$. We also assume that the a_{ij} satisfy the standard elliptic property

$$\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \ge d\|\xi\|^2,$$

where $d > 0$ is constant and $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^n$.

The operator (10.10) has an associated bilinear form

$$B(u,u) = \langle Lu, u\rangle = \langle u, L^*u\rangle, \tag{7.11}$$

where $\langle \cdot, \cdot \rangle$ is the $L^2(\mathcal{M})$ inner product and L^* is the L^2-adjoint defined by

$$L^*u = -\sum_{i,j=1}^{n} (a_{ij}(x)u_{x_i})_{x_j} - \sum_{i=1}^{n} (b_i(x)u)_{x_i} + c(x)u. \tag{7.12}$$

Using the bilinear form $B(u,u)$, the elliptic operator (10.10) defines an elliptic operator

$$L : L^2(\mathcal{M}) \to L^2(\mathcal{M}), \quad \text{with domain of definition } D(L) = H^2(\mathcal{M}). \tag{7.13}$$

It is a standard argument in linear elliptic PDE to show that there exists a $c > 0$ such the operator $L + cI : H^2(\mathcal{M}) \to L^2(\mathcal{M})$ is bounded and bijective. In particular, one shows that there exists a $c > 0$ such that the associated bilinear form $B(u,u) + c\|u\|_2$ is coercive and then applies the Lax-Milgram Theorem to conclude that there exists a unique weak solution $u \in H^1(\mathcal{M})$ to

$$Lu - cu = f \quad \text{for every } f \in L^2(\mathcal{M}).$$

Standard elliptic regularity theory implies that $u \in H^2(\mathcal{M})$ and the norm $\|\cdot\|_{2,2}$ makes $D(L)$ a Hilbert space. An application of the Open Mapping Theorem (Bounded Inverse Theorem) then implies that

$$(L + cI)^{-1} : L^2(\mathcal{M}) \to D(L),$$

is continuous. This implies that the operator $L + cI$ is closed and that the operator $(L + cI) - cI = L$ is closed. In addition, the operator

$$K_c = (L + cI)^{-1} \in L(L^2(\mathcal{M}), L^2(\mathcal{M})) \quad \text{is compact}$$

given that the embedding $H^2(\mathcal{M}) \subset L^2(\mathcal{M})$ is compact. For $f \in L^2(\mathcal{M})$, we have the equivalence

$$Lu = f, \quad u \in H^2(\mathcal{M}) \Leftrightarrow \tag{7.14}$$

$$u - cK_c u = K_c f, \quad u \in L^2(\mathcal{M}). \tag{7.15}$$

Riesz-Schauder theory implies that $(I - cK_c)$ is a Fredholm operator and the equivalence (10.14) implies that L is a Fredholm operator.

Because L is a Fredholm operator of index zero, we have that $R(L)$ is closed. Therefore we may write

$$L^2(\mathcal{M}) = R(L) \oplus Z_0,$$

where $Z_0 = R(L)^\perp$ is the orthogonal complement with respect to the L^2-inner product. Because $D(L)$ is dense in $L^2(\mathcal{M})$ and L is closed, may apply the Closed Range Theorem to conclude that

$$R(L) = \{f \in L^2(\mathcal{M}) \mid \langle f, u \rangle = 0 \quad \text{for all } u \in N(L^*)\} \tag{7.16}$$

and that $Z_0 = N(L^*)$, where $L^* : L^2(\mathcal{M}) \to L^2(\mathcal{M})$ is induced by (10.12). Therefore

$$L^2(\mathcal{M}) = R(L) \oplus N(L^*),$$

and if $D(L^*) = H^2(\mathcal{M})$, the above arguments imply that L^* is Fredholm operator. So we have the following decomposition of the codomain of L^*:

$$L^2(\mathcal{M}) = R(L^*) \oplus N(L). \tag{7.17}$$

Finally, given that $N(L) \subset D(L) = H^2(\mathcal{M}) \subset L^2(\mathcal{M})$, the decomposition (10.17) allows us to obtain the following Liapunov-Schmidt decomposition for the linear problem $L : H^2(\mathcal{M}) \to L^2(\mathcal{M})$:

$$H^2(\mathcal{M}) = N(L) \oplus (R(L^*) \cap H^2(\mathcal{M})), \tag{7.18}$$

$$L^2(\mathcal{M}) = R(L) \oplus N(L^*). \tag{7.19}$$

Now we observe that the Fredholm properties of linear elliptic operators derived on Hilbert spaces hold for subspaces that are only Banach spaces. We then use these Fredholm properties to derive Liapunov-Schmidt decompositions for these Banach spaces.

Suppose that the Banach space $Z \subset L^2(\mathcal{M})$ is continuously embedded and that the domain of definition $X \subset Z$ with a given norm is a Banach space that satisfies the following conditions:

$$L : X \to Z \quad \text{is continuous ,} \tag{7.20}$$
$$Lu = f \quad \text{for } u \in D(L) = H^2(\mathcal{M}), f \in Z \Rightarrow u \in X.$$

Equation (10.20) is an elliptic regularity condition and is satisfied for a variety of spaces, most notably $X = W^{2,p}(\mathcal{M}), Z = L^p(\mathcal{M})$ and $X = C^{2,\alpha}(\mathcal{M}), Z = C^{0,\alpha}(\mathcal{M})$ with the standard norms. Then for X and Z satisfying (10.18) and (10.20) we have that

$$N(L) = N(L|_Z) \subset X, \quad \text{and} \tag{7.21}$$
$$R(L) \cap Z = R(L|_Z) \quad \text{is closed in } Z, \tag{7.22}$$

given that $Z \subset L^2(\mathcal{M})$ is continuously embedded and $R(L)$ is closed in $L^2(\mathcal{M})$. The ellipticity property (10.20) also holds for the adjoint L^* and implies that

$$N(L^*) \subset X, \quad \text{where } D(L^*) = D(L) = X.$$

Applying the decomposition (10.19), we may write any $z \in Z$ as

$$z = Lu + u^*, \quad \text{where } u \in D(L), u^* \in N(L^*), \tag{7.23}$$
$$Lu = z - u^* \in Z \Rightarrow u \in X, \quad \text{therefore}$$
$$Z = R(L|_Z) \oplus N(L^*).$$

Finally, we have that $\dim N(L|_Z) = \dim N(L) = \dim N(L^*)$ and that

$$L : X \to Z, \ X = D(L|_Z), \text{ is a Fredholm operator of index zero.} \tag{7.24}$$

The decomposition (10.18) then implies that

$$X = N(L|_Z) \oplus (R(L^*) \cap X), \tag{7.25}$$

and so (10.23) and (10.25) constitute a Liapunov-Schmidt decomposition of the spaces X and Z with respect to a given linear, elliptic operator L.

Remark 7.8. *As noted in* [11]*, we may regard the spaces $W^{2,p}(\mathcal{M}) \subset L^p(\mathcal{M}) \subset L^2(\mathcal{M})$ for $p > 2$, and we can then apply the above discussion to conclude that a linear elliptic operator $L : W^{2,p}(\mathcal{M}) \to L^p(\mathcal{M})$ is Fredholm and use this fact to obtain a Liapunov-Schmidt decomposition of $X = W^{2,p}(\mathcal{M})$ and $Z = L^p(\mathcal{M})$. Similarly, $C^{2,\alpha}(\mathcal{M}) \subset C^{0,\alpha}(\mathcal{M}) \subset L^2(\mathcal{M})$ for $\alpha \in (0,1)$, so $L : C^{2,\alpha}(\mathcal{M}) \to C^{0,\alpha}(\mathcal{M})$ is Fredholm and we may also obtain a Liapunov-Schmidt decomposition of $X = C^{2,\alpha}(\mathcal{M})$ and $Z = C^{0,\alpha}(\mathcal{M})$ using* (10.23) *and* (10.25)*.*

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E-mail address: mholst@math.ucsd.edu

E-mail address: meiercaleb@gmail.com

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA SAN DIEGO, LA JOLLA CA 92093