MATH 31AH, LINEAR ALGEBRA, EXAM 1

Friday, October 23rd, 2015, 1-1:50pm, CSB 001

- *Your Name*: SOLUTIONS.
- *ID Number*:
- *Section (circle)*: A01 (2:00 PM) A02 (3:00 PM)

Problem 1. Let $x = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$ and $y = \begin{pmatrix} -1 \\ 0 \\ 2 \\ 1 \end{pmatrix}$.

(a) Find $x + y$ and $2x$.

(b) Compute their lengths $\|x\|$ and $\|y\|$, and the dot product $x \bullet y$.

(c) Calculate the orthogonal projection of x onto the line spanned by y.

(a) $x+y = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ and $2x = \begin{pmatrix} 2 \\ 0 \\ -2 \\ 0 \end{pmatrix}$.

(b) $\|x\| = \sqrt{2}$, $\|y\| = \sqrt{6}$, $x \bullet y = -3$.

(c) $\mathrm{proj}(x) = \dfrac{x \bullet y}{\|y\|^2} y = \dfrac{-3}{6} \begin{pmatrix} -1 \\ 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 0 \\ -1 \\ -1/2 \end{pmatrix}$.

Problem 2. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by

$$T(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}) = \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

(a) Find the three vectors $T(e_1)$, $T(e_2)$, and $T(e_3)$.

(b) Evaluate $T(\begin{pmatrix} 5 \\ 7 \\ -1 \end{pmatrix})$.

(c) Is T injective? Is T surjective?

(a) $T(e_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $T(e_2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $T(e_3) = \begin{pmatrix} 5 \\ 7 \end{pmatrix}$.

(b) $T(\begin{pmatrix} 5 \\ 7 \\ -1 \end{pmatrix}) = \begin{pmatrix} 1\cdot 5 + 0\cdot 7 + 5\cdot(-1) \\ 0\cdot 5 + 1\cdot 7 + 7\cdot(-1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

(c) T is <u>not</u> injective:

$\begin{pmatrix} 5 \\ 7 \\ -1 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, but $T(\begin{pmatrix} 5 \\ 7 \\ -1 \end{pmatrix}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = T(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix})$.

T <u>is</u> surjective: $\forall x_1, x_2$.

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = T(\begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix})$. is in the image of T.

Problem 3. Let $A = \begin{pmatrix} x & 1 \\ 2 & 3 \end{pmatrix}$ where x can be any real number.

(a) Write down its transpose A^T.

(b) For which values of x does the equality $A^T A = AA^T$ hold?

(c) For which values of x is A invertible? For such x, write down A^{-1}.

(a) $A^T = \begin{pmatrix} x & 2 \\ 1 & 3 \end{pmatrix}$

(b) $A^T A = \begin{pmatrix} x & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} x^2+4 & x+6 \\ x+6 & \underline{10} \end{pmatrix}$

$AA^T = \begin{pmatrix} x & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} x^2+1 & 2x+3 \\ 2x+3 & \underline{13} \end{pmatrix}$.

Never the same: $\underline{\text{No}}$ x.

(c) A invertible $\iff \det(A) \neq 0$.

$\det(A) = x \cdot 3 - 1 \cdot 2 = 3x - 2$,

which is $\underline{\text{nonzero}}$ for all $x \neq \frac{2}{3}$.

For such x,

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} 3 & -1 \\ -2 & x \end{pmatrix} = \frac{1}{3x-2} \cdot \begin{pmatrix} 3 & -1 \\ -2 & x \end{pmatrix}.$$

Problem 4. Introduce two matrices $A = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -1 \\ 0 & 3 \\ 2 & 1 \end{pmatrix}$.

(a) Calculate the two matrix products AB and BA.

(b) Find the matrix $A^T AB(AB)^{-1}B^T$.

(c) Verify that $\{x \in \mathbb{R}^3 : Ax = 0\}$ is a line through the origin in $\mathbb{R}^3$.

(a) $AB = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 4 \\ 3 & 6 \end{pmatrix}$ (invertible: $\det = -18 \neq 0$)

$BA = \begin{pmatrix} 1 & -1 \\ 0 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} -2 & -1 & -1 \\ 3 & 6 & 3 \\ -1 & 4 & 1 \end{pmatrix}.$

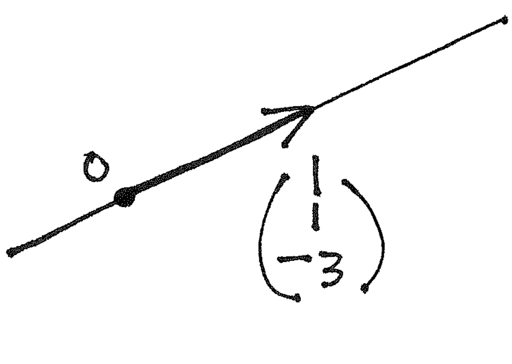
(b) $A^T \underbrace{(AB)(AB)^{-1}}_{= I} B^T = A^T B^T = (BA)^T = \begin{pmatrix} -2 & 3 & -1 \\ -1 & 6 & 4 \\ -1 & 3 & 1 \end{pmatrix}.$

(c) $Ax = \begin{pmatrix} -x_1 + x_2 \\ x_1 + 2x_2 + x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ means $\begin{cases} x_1 = x_2 \text{ and} \\ x_1 + 2x_2 + x_3 = 0. \end{cases}$

- Let $x_1 = x_2 = t$, and substitute into the 2nd eqn.: $x_3 = -3t$.

Hence,

$\{x \in \mathbb{R}^3 : Ax = 0\} = \left\{ t \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} : t \in \mathbb{R} \right\}$, which is the line in $\mathbb{R}^3$ (through 0) spanned by $\begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix}$.



Problem 5. Which of the following subsets of $\mathbb{R}^4$ are subspaces? Explain.

✓ (a) $V_1 = \{x \in \mathbb{R}^4 : \|x\| = 0\}$.

(b) $V_2 = \{x \in \mathbb{R}^4 : x_1 + 2x_2 + 3x_3 + 4x_4 = \underline{5}\}$.

(c) $V_3 = \{x \in \mathbb{R}^4 : x_1 x_3 \geq 0\}$.

✓ (d) $V_4 = \{x \in \mathbb{R}^4 : x_1^2 + x_3^2 \geq 0\}$.

✓ (e) $V_5 = \{x \in \mathbb{R}^4 : x_1 = x_3 \text{ and } x_2 = -x_4\}$.

(a) $V_1 = \{0\}$ <u>is</u> a ("trivial") subspace.

(b) V_2 is <u>not</u>: The zero vector doesn't satisfy the (inhomogeneous) eqn.

(c) V_3 is <u>not</u>: $\begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 0 \\ -1 \\ 0 \end{pmatrix}$ both lie in V_3, but their sum $\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ doesn't.

(d) V_4 <u>is</u> a ("trivial") subspace; the condition is always fulfilled.
$\searrow \mathbb{R}^4$

(e) V_5 <u>is</u> a ("non-trivial") subspace: It's the space of all linear combinations of

$$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

~ <u>plane</u> in $\mathbb{R}^4$.