Math 31AH, Linear Algebra, Exam 2

Friday, November 20th, 2015, 2-2:50pm, CSB 001

- *Your Name*: SOLUTIONS
- *ID Number*:
- *Section (circle)*: A01 (2:00 PM) A02 (3:00 PM)

Problem 1. Consider the three vectors in $\mathbb{R}^3$ given below.

$$v_1 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 5 \\ -8 \\ -5 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}.$$

(a) Show that $\{v_1, v_2, v_3\}$ is a basis for $\mathbb{R}^3$.

(b) Find the volume of the parallelepiped spanned by $\{v_1, v_2, v_3\}$.

$A = (v_1 v_2 v_3) =$

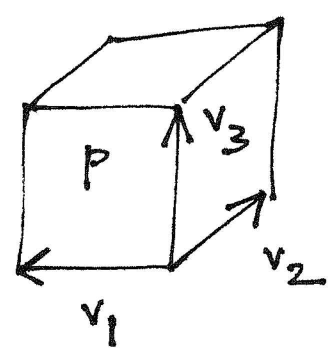
(a) $$\underbrace{\begin{pmatrix} -1 & 5 & 2 \\ 2 & -8 & 3 \\ 1 & -5 & 1 \end{pmatrix}}_{\text{Asking: invertible?}} \sim \begin{pmatrix} -1 & 5 & 2 \\ 0 & 2 & 7 \\ 0 & 0 & 3 \end{pmatrix},$$ using only row addition.

triangular.

Therefore, $\det(A) = (-1)\cdot 2 \cdot 3 = -6$, which is nonzero.

(b) $P = \{ x_1 v_1 + x_2 v_2 + x_3 v_3 \mid 0 \leq x_1, x_2, x_3 \leq 1 \}$ has volume:

$$\mathrm{vol}(P) = |\det(A)| = |-6| = \boxed{6}$$



Problem 2. Let $V \subset \mathbb{R}^4$ be the set of vectors perpendicular to both

$$u = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \\ 3 \\ 4 \end{pmatrix} = v$$

(a) Explain why V is a subspace of $\mathbb{R}^4$.

(b) Find a basis for V.

(c) What is the dimension of V?

(a) Let $A = \begin{pmatrix} \text{— } u \text{ —} \\ \text{— } v \text{ —} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 3 & 4 \end{pmatrix}$.

Note: $x \in V$ means $\begin{cases} x \cdot u = 0 \\ x \cdot v = 0 \end{cases}$ i.e., $Ax = 0$.

~ Then: $V = \{x \in \mathbb{R}^4 : Ax = 0\}$ (the "null space" of A)

In particular, V is a subspace.

(b) A is already in echelon form, so to solve $Ax = 0$ identify the free variables: $x_3 = s$, $x_4 = t$.

~ Then solve for x_1, x_2: $x_1 = -s - 2t$, $x_2 = -3s - 4t$.

Altogether:

$$V = \left\{ s\begin{pmatrix} -1 \\ -3 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -2 \\ -4 \\ 0 \\ 1 \end{pmatrix} : s, t \in \mathbb{R} \right\}.$$

$\Rightarrow$ Conclude: $\left\{ \begin{pmatrix} -1 \\ -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ -4 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis for V.

(c) $\dim(V) =$ # vectors in a basis $= \boxed{2}$.

Problem 3. Let $A = \begin{pmatrix} 3 & -1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{pmatrix}$.

(a) Is A invertible? If so, find the inverse matrix A^{-1}.

(b) Write $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ as a linear combination of the columns of A.

(c) Compute the determinant of AA^T.

(a) Do both at the same time: $(A|I) = \left(\begin{array}{ccc|ccc} 3 & -1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \sim$

$$\left(\begin{array}{ccc|ccc} 3 & -1 & 0 & 1 & 0 & -1 \\ 0 & 2 & 0 & 0 & 1 & -4 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 3 & -1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \sim$$

$$\left(\begin{array}{ccc|ccc} 3 & 0 & 0 & 1 & \frac{1}{2} & -3 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & \frac{1}{6} & -1 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) = (I|A^{-1}).$$

Since $A \sim I$, A <u>is</u> invertible, and

$$\underline{\underline{A^{-1} = \frac{1}{6}\begin{pmatrix} 2 & 1 & -6 \\ 0 & 3 & -12 \\ 0 & 0 & 6 \end{pmatrix}}}.$$

(b) $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = x_1 \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}$

$$= Ax, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A^{-1}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} -5 \\ -9 \\ 6 \end{pmatrix} = \underline{\underline{\begin{pmatrix} -5/6 \\ -3/2 \\ 1 \end{pmatrix}}}.$$

<u>Conclude</u>: $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \frac{-5}{6}\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + \frac{-3}{2}\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}$.

(c) $\det(AA^T) \underset{\text{product formula}}{=} \det(A)\det(A^T) = \det(A)^2 = 6^2 = \boxed{36}$

product formula

$\det(A) = \det(A^T) = 6 = 3\cdot 2\cdot 1$.
(since A is <u>triangular</u>)

Problem 4. In $\mathbb{R}^4$ we consider the three vectors below.

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

(a) Are they linearly independent?

(b) For which value of x does $\begin{pmatrix} 1 \\ 0 \\ 0 \\ x \end{pmatrix}$ belong to $\operatorname{span}(v_1, v_2, v_3)$?

no free variables: (pivot in every column).

(a) $A = (v_1 v_2 v_3) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \tilde{A}$

– which shows $Ax = 0$ has only $x = 0$ as solution. I.e., ~~yes~~, $\{v_1, v_2, v_3\}$ are linearly independent.

(b) Asking when (which x)

$$(*) \quad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ x \end{pmatrix}$$

is consistent:

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & x \end{array}\right) \sim$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & x-1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & -1 & x-1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 & 1-x \end{array}\right) \sim$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2}-x \end{array}\right) \leftarrow$$

Looking at the last row, $(*)$ has a solution precisely when $\frac{1}{2} - x = 0$. I.e.,

$$\boxed{x = \frac{1}{2}}$$

(in which case the solution is: $x_1 = \frac{1}{2}$, $x_2 = -\frac{1}{2}$, $x_3 = \frac{1}{2}$)