Coordinates and Change of Basis

Let $\mathcal{B} = [\mathbf{b}_1, \ldots, \mathbf{b}_n]$ be an ordered basis for the n-dimensional vector space V.

If $\mathbf{v}$ is any vector in V, then there is a unique coordinate vector $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ in $\mathbb{R}^n$ such that $\mathbf{v} = x_1\mathbf{b}_1 + \cdots + x_n\mathbf{b}_n = [\mathbf{b}_1, \ldots, \mathbf{b}_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \mathcal{B}[\mathbf{v}]_{\mathcal{B}}$.

If $\mathcal{D} = [\mathbf{d}_1, \ldots, \mathbf{d}_n]$ is another ordered basis for V, then there are unique scalars P_{ij} such that $\mathbf{d}_j = \sum_{i=1}^{n} \mathbf{b}_i P_{ij}$, for $1 \le j \le n$. In other words,

$$[\mathbf{d}_1, \ldots, \mathbf{d}_n] = [\mathbf{b}_1, \ldots, \mathbf{b}_n] \begin{bmatrix} P_{11} & \cdots & P_{1n} \\ \vdots & & \vdots \\ P_{n1} & \cdots & P_{nn} \end{bmatrix}.$$

Thus $\mathcal{D} = \mathcal{B}P$, where P is the matrix (P_{ij}).

Let $[\mathbf{v}]_{\mathcal{D}} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$. Then $\mathbf{v} = y_1\mathbf{d}_1 + \cdots + y_n\mathbf{d}_n = [\mathbf{d}_1, \ldots, \mathbf{d}_n] \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \mathcal{D}[\mathbf{v}]_{\mathcal{D}}$.

Since $\mathbf{v}$ is also equal to $\mathcal{B}[\mathbf{v}]_{\mathcal{B}}$, we have $\mathcal{D}[\mathbf{v}]_{\mathcal{D}} = \mathcal{B}[\mathbf{v}]_{\mathcal{B}}$, and since $\mathcal{D} = \mathcal{B}P$, it follows that $\mathcal{B}P[\mathbf{v}]_{\mathcal{D}} = \mathcal{B}[\mathbf{v}]_{\mathcal{B}}$.

Since $\mathcal{B}$ is a basis for V, $\mathbf{v}$ is *uniquely* represented as a linear combination of elements of $\mathcal{B}$. This means that

$$[\mathbf{v}]_{\mathcal{B}} = P[\mathbf{v}]_{\mathcal{D}}.$$

Similarly,

$$[\mathbf{v}]_{\mathcal{D}} = P^{-1}[\mathbf{v}]_{\mathcal{B}}.$$

P is called the *transition matrix* from the ordered basis $\mathcal{D}$ to the ordered basis $\mathcal{B}$.

Note: In the notation of **Proposition and definition 2.6.18** on page 217 of your book,

$$P = [P_{\mathcal{D}\to\mathcal{B}}].$$

It's also worth observing that

$$P^{-1} = [P_{\mathcal{D}\to\mathcal{B}}]^{-1} = [P_{\mathcal{B}\to\mathcal{D}}].$$