Math 31AH
Dimension of a Vector Space

Theorem. *If a vector space V has a basis of n vectors, then every basis of V must consist of exactly n vectors.*

Proof. Let $\mathcal{V} = \{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ and $\mathcal{W} = \{\mathbf{w}_1, \ldots, \mathbf{w}_p\}$ be finite indexed subsets of a vector space V. Suppose that

1. $\mathcal{V}$ spans V; that is, $V = \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$, and
2. $\mathcal{W}$ is linearly independent; that is, the vector equation $x_1\mathbf{w}_1 + \cdots + x_p\mathbf{w}_p = \mathbf{0}$ has only the trivial solution $x_1 = \cdots = x_p = 0$.

Then, the set $\{\mathbf{w}_p, \mathbf{v}_1, \ldots, \mathbf{v}_m\}$, obtained by adjoining $\mathbf{w}_p$ to the beginning of the indexed list of vectors in $\mathcal{V}$, is linearly dependent since $\mathcal{V} = \{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ spans V. Thus, at least one of the vectors in this set, say $\mathbf{v}_k$, is a linear combination of the preceding vectors $\mathbf{w}_p, \mathbf{v}_1, \ldots, \mathbf{v}_{k-1}$. It follows that the set $\{\mathbf{w}_p, \mathbf{v}_1, \ldots, \mathbf{v}_{k-1}, \mathbf{v}_{k+1}, \ldots, \mathbf{v}_m\}$, obtained by removing $\mathbf{v}_k$, still spans V. Renumbering the $\mathbf{v}$'s, we obtain a set $S_1 = \{\mathbf{w}_p, \mathbf{v}_1, \ldots, \mathbf{v}_{m-1}\}$ which spans V.

Similarly, the set $\{\mathbf{w}_{p-1}, \mathbf{w}_p, \mathbf{v}_1, \ldots, \mathbf{v}_{m-1}\}$, obtained by adjoining $\mathbf{w}_{p-1}$ to the beginning of the indexed list of vectors in S_1, is linearly dependent since S_1 spans V. Thus, one of these vectors is a linear combination of the previous vectors, and since $\{\mathbf{w}_{p-1}, \mathbf{w}_p\}$ is linearly independent, it must be one of the $\mathbf{v}$'s that is a linear combination of the previous vectors. After removing this vector and renumbering the $\mathbf{v}$'s (if necessary), we obtain a set $S_2 = \{\mathbf{w}_{p-1}, \mathbf{w}_p, \mathbf{v}_1, \ldots, \mathbf{v}_{m-2}\}$ which spans V.

Continuing in this fashion, we obtain a sequence of sets which span V:

$$\begin{array}{rcl} S_1 & = & \{\mathbf{w}_p, \mathbf{v}_1, \ldots, \mathbf{v}_{m-1}\} \\ S_2 & = & \{\mathbf{w}_{p-1}, \mathbf{w}_p, \mathbf{v}_1, \ldots, \mathbf{v}_{m-2}\} \\ \cdots & & \cdots \\ S_k & = & \{\mathbf{w}_{p-k+1}, \ldots, \mathbf{w}_p, \mathbf{v}_1, \ldots, \mathbf{v}_{m-k}\} \\ \cdots & & \cdots \\ S_p & = & \{\mathbf{w}_1, \ldots, \mathbf{w}_p, \mathbf{v}_1, \ldots, \mathbf{v}_{m-p}\} \end{array}$$

where $S_p = \{\mathbf{w}_1, \ldots, \mathbf{w}_p\}$ if $p = m$.

Note that p cannot be greater than m. For, if it were, we would obtain a set $S_p = \{\mathbf{w}_{p-m+1}, \ldots, \mathbf{w}_p\}$ that spans V. Then, the set $\{\mathbf{w}_{p-m}, \mathbf{w}_{p-m+1}, \ldots, \mathbf{w}_p\}$ would be linearly *dependent*, contrary to our initial assumption. Therefore, we conclude that $p \leq m$; that is, the number of vectors in any linearly independent subset of V is less than or equal to the number of vectors in a subset of V that spans V.

If both $\mathcal{V}$ and $\mathcal{W}$ are bases for V; that is, that they both are linearly independent and span V, then the above argument can be applied to $\mathcal{V}$ and $\mathcal{W}$ with their roles reversed ($\mathcal{W}$ spans V and $\mathcal{V}$ is linearly independent) to conclude that $m \leq p$. It follows that $p = m$. $\square$