Matrix Representation of Linear Transformations

Let V be an n-dimensional vector space with ordered basis $\mathcal{B} = [\mathbf{b}_1, \ldots, \mathbf{b}_n]$, let W be an m-dimensional vector space with ordered basis $\mathcal{D} = [\mathbf{d}_1, \ldots, \mathbf{d}_m]$, and let $T : V \to W$ be a linear transformation. We know that for each j between 1 and n,

$$T(\mathbf{b}_j) = t_{1j}\mathbf{d}_1 + \cdots + t_{mj}\mathbf{d}_m = [\mathbf{d}_1, \ldots, \mathbf{d}_m] \begin{bmatrix} t_{1j} \\ \vdots \\ t_{mj} \end{bmatrix}$$

for some choice of scalars $t_{1j}, \ldots, t_{mj}$, since $\mathcal{D} = [\mathbf{d}_1, \ldots, \mathbf{d}_m]$ is a basis for W.

Therefore,

$$\begin{aligned} [T(\mathbf{b}_1), \ldots, T(\mathbf{b}_n)] &= [t_{11}\mathbf{d}_1 + \cdots + t_{m1}\mathbf{d}_m, \quad \ldots \quad, t_{1n}\mathbf{d}_1 + \cdots + t_{mn}\mathbf{d}_m] \\ &= [\mathbf{d}_1, \ldots, \mathbf{d}_m] \begin{bmatrix} t_{11} & \cdots & t_{1n} \\ \vdots & & \vdots \\ t_{m1} & \cdots & t_{mn} \end{bmatrix}. \end{aligned}$$

$\begin{bmatrix} t_{11} & \cdots & t_{1n} \\ \vdots & & \vdots \\ t_{m1} & \cdots & t_{mn} \end{bmatrix}$ is called the *matrix representing T relative to the bases $\mathcal{B}$, $\mathcal{D}$*, and is denoted by $[T]_{\mathcal{D}\mathcal{B}}$ (note the order).

We have shown that if V and W are vector spaces with ordered bases $\mathcal{B}$ and $\mathcal{D}$, and if $T : V \to W$ is a linear transformation, then

$$[T(\mathbf{b}_1), \ldots, T(\mathbf{b}_n)] = [\mathbf{d}_1, \ldots, \mathbf{d}_m] \begin{bmatrix} t_{11} & \cdots & t_{1n} \\ \vdots & & \vdots \\ t_{m1} & \cdots & t_{mn} \end{bmatrix} = \mathcal{D}\,[T]_{\mathcal{D}\mathcal{B}},$$

where $[T]_{\mathcal{D}\mathcal{B}}$ is the matrix representing T with respect to the ordered bases $\mathcal{B}$, $\mathcal{D}$.

Let $\mathbf{x}$ be a vector in V and let $\mathbf{y} = T(\mathbf{x})$. Since $\mathbf{y} = T(\mathbf{x})$ is a vector in W,

$$T(\mathbf{x}) = y_1\mathbf{d}_1 + \cdots + y_m\mathbf{d}_m = [\mathbf{d}_1, \ldots, \mathbf{d}_m] \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} = \mathcal{D}\,[\mathbf{y}]_{\mathcal{D}} = \mathcal{D}\,[T(\mathbf{x})]_{\mathcal{D}}.$$

On the other hand,

$$\begin{aligned} T(\mathbf{x}) &= T(x_1\mathbf{b}_1 + \cdots + x_n\mathbf{b}_n) \\ &= x_1 T(\mathbf{b}_1) + \cdots + x_n T(\mathbf{b}_n) \\ &= [T(\mathbf{b}_1), \ldots, T(\mathbf{b}_n)] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \\ &= \mathcal{D}\,[T]_{\mathcal{D}\mathcal{B}}\,[\mathbf{x}]_{\mathcal{B}}. \end{aligned}$$

Therefore, $\mathcal{D}\,[T(\mathbf{x})]_{\mathcal{D}} = \mathcal{D}\,[T]_{\mathcal{D}\mathcal{B}}\,[\mathbf{x}]_{\mathcal{B}}$. In other words, $[T(\mathbf{x})]_{\mathcal{D}} = [T]_{\mathcal{D}\mathcal{B}}\,[\mathbf{x}]_{\mathcal{B}}$.

If $T : V \to V$ and $\mathcal{D} = \mathcal{B}$, we write $[T]_{\mathcal{B}}$ rather than $[T]_{\mathcal{B}\mathcal{B}}$.