Transition Matrix Exercises

1. Recall the double-angle trigonometric identities.

$$\begin{aligned} \cos^2(x) - \sin^2(x) &= \cos(2x). \\ 2\sin(x)\cos(x) &= \sin(2x). \end{aligned}$$

   (a) Show that $\mathcal{S} = \{\sin^2(x), \cos^2(x), \sin(x)\cos(x)\}$ and $\mathcal{T} = \{1, \sin(2x), \cos(2x)\}$ span the same 3-dimensional subspace of $C[0, \pi] = \{f : [0, \pi] \to \mathbb{R} \mid f \text{ is continuous}\}$.

   (b) Find the transition matrix from the ordered basis $\mathcal{S}$ to the ordered basis $\mathcal{T}$.

   (c) Use the transition matrix to express $a\sin^2(x) + b\cos^2(x)$ as a linear combination of 1, $\sin(2x)$ and $\cos(2x)$.

2. Recall that the hyperbolic cosine and hyperbolic sine functions are defined as follows.

$$\begin{aligned} \cosh(x) &= \frac{e^x + e^{-x}}{2}. \\ \sinh(x) &= \frac{e^x - e^{-x}}{2}. \end{aligned}$$

   (a) Show that $\mathcal{E} = \{e^x, e^{-x}\}$ and $\mathcal{H} = \{\cosh(x), \sinh(x)\}$ span the same 2-dimensional subspace of $C(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous}\}$.

   (b) Find the transition matrix from the ordered basis $\mathcal{E}$ to the ordered basis $\mathcal{H}$.

   (c) Use the transition matrix to express $ce^x + de^{-x}$ as a linear combination of $\sinh(x)$ and $\cosh(x)$.

3. Using the definition of the hyperbolic cosine function and the hyperbolic sine function, it is a straightforward computation to verify the following identities.

$$\begin{aligned} \cosh^2(x) - \sinh^2(x) &= 1. \\ \cosh^2(x) + \sinh^2(x) &= \cosh(2x). \\ 2\cosh(x)\sinh(x) &= \sinh(2x). \end{aligned}$$

   (a) Show that $\mathcal{S}_h = \{\cosh^2(x), \sinh^2(x), \cosh(x)\sinh(x)\}$ and $\mathcal{T}_h = \{1, \cosh(2x), \sinh(2x)\}$ span the same 3-dimensional subspace of $C(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous}\}$.

   (b) Find the transition matrix from the ordered basis $\mathcal{S}_h$ to the ordered basis $\mathcal{T}_h$.

   (c) Use the transition matrix to express $a\cosh^2(x) + b\sinh^2(x)$ as a linear combination of 1, $\cosh(2x)$ and $\sinh(2x)$.

Answers

1. (b) $P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}.$

   (c) $a\sin^2(x) + b\cos^2(x) = \frac{1}{2}(a+b) + \frac{1}{2}(-a+b)\cos(2x).$

2. (b) $P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$

   (c) $ce^x + de^x = (c+d)\cosh(x) + (c-d)\sinh(x).$

3. (b) $P = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}.$

   (c) $a\cosh^2(x) + b\sinh^2(x) = \frac{1}{2}(a-b) + \frac{1}{2}(a+b)\cosh(2x).$