MATH 110–003
Winter 2018
Solution to Practice problems 1

Section 1.4: The Tangent Problem

1. Let f be the function defined by $f(x) = 4x^2$. Let x be different from 3. What is the slope m_x of the line through the points $(3, 36)$ and $(x, 4x^2)$? Simplify your answer as much as possible.

   Solution. First recall that the slope of a line through $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is given by the formula
   $$\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1}.$$
   Let $x \neq 3$. The slope of the line through $(3, 36)$ and $(x, 4x^2)$ is then
   $$m_x = \frac{4x^2 - 36}{x - 3}.$$
   To simply it, we need to factor the numerator and the denominator. The numerator is the difference of two squares as $4x^2 - 36 = (2x)^2 - (6)^2$. So by the formula $a^2 - b^2 = (a - b)(a + b)$, we have $4x^2 - 36 = (2x - 6)(2x + 6)$. The denominator is already on the factored form. Thus
   $$m_x = \frac{(2x - 6)(2x + 6)}{x - 3} = \frac{2(x - 3)2(x + 3)}{x - 3} = 4(x + 3).$$
   We have simplified by $x - 3$ because $x \neq 3$, so that $x - 3 \neq 0$.

2. Let f be the function defined by $f(x) = \frac{2}{3x}$. Let x be different from 0 and 1. What is the slope m_x of the line through the points $(1, \frac{2}{3})$ and $(x, \frac{2}{3x})$? Simplify your answer as much as possible.

   Solution. First recall some algebra about fractions.
   $$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}, \frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}, \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b}\frac{d}{c} = \frac{ad}{bc}, \frac{\frac{a}{b}}{c} = \frac{a}{bc}, \frac{a}{\frac{b}{c}} = \frac{ac}{b}.$$
   Let x be different from 0 and 1. The slope of the line through $(1, \frac{2}{3})$ and $(x, \frac{2}{3x})$ is
   $$m_x = \frac{\frac{2}{3x} - \frac{2}{3}}{x - 1}.$$
   To simplify this, we first need to make the denominators of $\frac{2}{3x} - \frac{2}{3}$ the same. By using $\frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd}$, we get $\frac{2}{3x} - \frac{2}{3} = \frac{6-6x}{9x}$. So
   $$m_x = \frac{\frac{6-6x}{9x}}{x - 1} = \frac{6 - 6x}{9x(x - 1)} = \frac{-6(x - 1)}{9x(x - 1)} = \frac{-6}{9x} = \frac{-2}{3x}.$$
   We have simplified by $x - 1$ because $x \neq 1$, so that $x - 1 \neq 0$.

3. The point $P(2, -1)$ lies on the curve $y = \frac{1}{1-x}$.

(a) If Q is the point $(x, \frac{1}{1-x})$, use your calculator to find the slope of the secant line PQ for the following values of x: $1.5, 1.9, 1.99, 1.999, 2.5, 2.1, 2.01, 2.001$.

(b) Using the results of part (a), guess the value of the slope of the tangent line to the curve at $P(2, -1)$.

(c) Using the slope from part (b), find an equation of the tangent line to the curve at $P(2, -1)$.

Solution. (a) For $x \neq 2$, the slope of the line through $P(2, -1)$ and $Q(x, \frac{1}{1-x})$ is

$$m_x = \frac{\frac{1}{1-x} - (-1)}{x-2} = \frac{\frac{1+(1-x)}{1-x}}{x-2} = \frac{2-x}{(1-x)(x-2)} = \frac{-(x-2)}{(1-x)(x-2)} = \frac{-1}{1-x}.$$

x	1.5	1.9	1.99	1.999	2	2.001	2.01	2.1	2.5
m_x	2	1.111	1.01	1.001		0.999	0.990	0.909	0.666

(b) From the table, we can make the guess that the slope of the the tangent line to the curve $y = \frac{1}{1-x}$ at $P(2, -1)$ is

$$m = \lim_{x \to 2} m_x = 1.$$

(c) An equation of the tangent line to the curve at $P(2, -1)$ is: $y - (-1) = m(x-2)$, which is equivalent to $y + 1 = 1(x-2)$, so that $y = x - 3$.

4. Let f be the function defined by $f(x) = -\frac{1}{x^2}$. Let x be different from 0 and 2.

(a) What is the slope m_x of the line through the points $(2, -\frac{1}{4})$ and $(x, -\frac{1}{x^2})$? Simplify your answer as much as possible.

(b) Guess the value of $\lim_{x \to 2} m_x$, and determine an equation for the line tangent to the graph of f at $(2, -\frac{1}{4})$.

Solution.

(a) Let $x \neq 0$ and $x \neq 2$. The slope of the line through the points $(2, -\frac{1}{4})$ and $(x, -\frac{1}{x^2})$ is

$$m_x = \frac{-\frac{1}{x^2} - (-\frac{1}{4})}{x-2} = \frac{-\frac{1}{x^2} + \frac{1}{4}}{x-2} = \frac{\frac{-4+x^2}{4x^2}}{x-2} = \frac{x^2-4}{4x^2(x-2)} = \frac{(x-2)(x+2)}{4x^2(x-2)} = \frac{x+2}{4x^2}.$$

(b) In order to guess the limit, we make a table.

x	1.5	1.9	1.99	1.999	2	2.001	2.01	2.1	2.5
m_x	0.388	0.27	0.2518	0.2501		0.2498	0.2481	0.232	0.18

From that table we can make the guess that $m = \lim_{x \to 2} m_x = 0.25 = \frac{1}{4}$. An equation of the tangent line is $y - (-\frac{1}{4}) = m(x-2)$, that is, $y + \frac{1}{4} = \frac{1}{4}(x-2)$ or $y = \frac{1}{4}x - \frac{3}{4}$.

Section 1.5: The Limit of a Function

1. Use the given graph of f (see Figure 1) to state the value of each quantity, if it exists. If it does not exist, explain why.

   (a) $\lim_{x\to 2^-} f(x)$; (b) $\lim_{x\to 2^+} f(x)$; (c) $\lim_{x\to 2} f(x)$; (d) $f(2)$; (e) $\lim_{x\to 4} f(x)$; (f) $f(4)$.

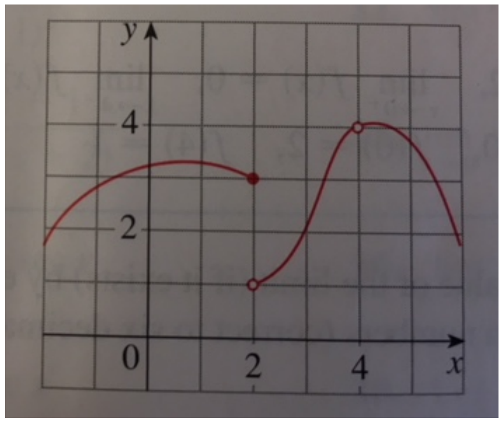

Figure 1

Solution. From the graph, we have: (a) $\lim_{x\to 2^-} f(x) = 3$; (b) $\lim_{x\to 2^+} f(x) = 1$; and (c) $\lim_{x\to 2} f(x)$ does not exist since the limit from the left is not the same as the limit from the right.

(d) $f(2) = 3$; (e) $\lim_{x\to 4} f(x) = 4$; and (f) $f(4)$ does not exist because of the hole.

2. For the function g whose graph is given (see Figure 2), state the value of each quantity, if it exists. If it does not exist, explain why.

   (a) $\lim_{x\to 0^-} g(t)$; (b) $\lim_{x\to 0^+} g(t)$; (c) $\lim_{x\to 0} g(t)$; (d) $\lim_{x\to 2^-} g(t)$; (e) $\lim_{x\to 2^+} g(t)$; (f) $\lim_{x\to 2} g(t)$; (g) $g(2)$; (h) $\lim_{x\to 4} g(t)$.

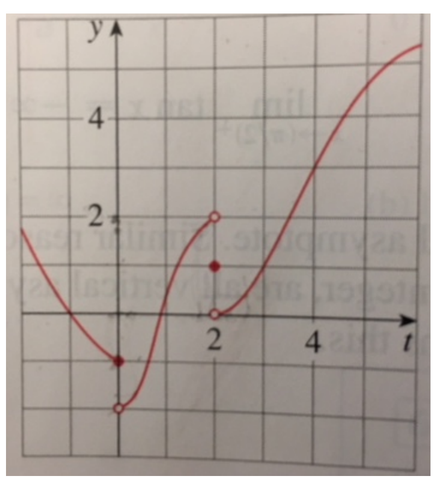

Figure 2

Solution. (a) $\lim_{t\to 0^-} g(t) = -1$; (b) $\lim_{t\to 0^+} g(t) = -2$; (c) $\lim_{t\to 0} g(t)$ does not exist (DNE) since the limit from the left is different from the limit from the right.

(d) $\lim_{t\to 2^-} g(t) = 2$; (e) $\lim_{t\to 2^+} g(t) = 0$; (f) $\lim_{t\to 2} g(t)$ DNE for the same reason as before.

(g) $g(2) = 1$ and $\lim_{t\to 4} g(t) = 3$.