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COMPLEXITY OF ALLOCATION PROCESSES: CHAOS AND PATH DEPENDENCE

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ABSTRACT

Allocation processes—the division of some commodity among multiple agents—are fundamental to social interactions in various arenas. Examples include wealth/income distribution in populations, natural resource exploitation, market share for competing corporations, satellite bandwidth division among many users, and CPU time usage by multiple software agents running simultaneously. In the case where each agent prefers more to less of the commodity—as in these examples—preference, or Condorcet, cycles are inevitable. We determine the consequences of this fact on an analytically tractable process of allocation subject to random external perturbations. This is a complex system: under majority rule the process is chaotic, while under weighted majority rule the system self-organizes to produce a path-dependent majority owner/dictator/monopolist.

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1. Introduction

EXAMPLE 1. Suppose Mom has baked an apple pie and is faced with the quintessential problem (no more American than are mothers or pies!) of allocating the pie among her children: Alice, Bob and Charlie. For reasons of her own she cuts a $\frac{1}{3}$ wedge of the pie and suggests to her children that she give it to Charlie and leave the remaining $\frac{2}{3}$ of the pie for Alice, with Bob getting none.

Bob complains vociferously, so she gives the three children an alternative: either stick with this initial allocation $a_1 = (\frac{2}{3}, 0, \frac{1}{3})$, or change to the allocation $a_2 = (0, \frac{1}{3}, \frac{2}{3})$, where these ordered triples indicate the amounts allocated to Alice, Bob and Charlie, respectively. Assuming each of the children prefers more to less of the pie, Bob and Charlie will vote to change to a_2; only Alice is worse off this way. But then Mom suggests the allocation $a_3 = (\frac{1}{3}, \frac{2}{3}, 0)$ and now Alice and Bob prefer this to a_2 so the result of a (majority) vote is to switch again.

Losing patience, Mom suggests $a_1 = (\frac{2}{3}, 0, \frac{1}{3})$ for the second time, and despite the fact that the children voted to switch from a_1 to a_2 and then to a_3, now they vote again by 2:1 to return to a_1, completing a cycle. Figure 1 illustrates this preference (Condorcet [1]) cycle: The three vertices represent the a_i while the directed edges indicate how decisions between two alternatives go when decided by majority vote.

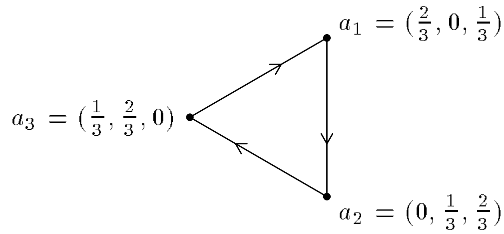

Figure 1. The directed graph f_p representing majority rule aggregation of the agents' more-is-better preferences contains a Condorcet cycle. The edges connecting each vertex to itself are omitted.

Under the very strict assumptions implicit in our story (*e.g.*, there are no side agreements between the children) this situation exemplifies a social system in which we attribute rationality to the individual agents (Alice, Bob and Charlie) in the form of totally ordered preferences [2] for allocations giving them bigger over smaller pie slices, yet cannot conclude that any equilibrium is achieved when these preferences are aggregated. In this paper we will explain the sense in which the absence of equilibria in allocation processes indicates that they are complex systems.

Let us begin by recalling a pragmatic definition of complexity [3]: A system is *complex* if it is represented efficiently by different models at different scales. That is, if a system can be simulated with bounded error and with less computational effort at larger scales in terms of macroscopic variables than in terms of microscopic variables, and the macroscopic variables differ qualitatively from the microscopic ones, then the system is complex. Table 1 summarizes some examples of complex systems for which the qualitative difference of the macroscopic model derives from nontrivial global topology [3].

The second and third systems in this table are essentially similar: there is a finite

SYSTEM	LOCAL MODEL	GLOBAL MODEL
fluid flow	scattering molecules	interacting vortices
population biology	interacting organisms	cycling ecology
market economics	agent utilities	market w/o stable equilibrium
iterated social choice	agent preferences	Condorcet cycles

Table 1. Four systems which can be complex ... when the global topology is nontrivial, producing the phenomena listed in the third column.

number of continuous valued variables in the local models—relative populations in the second and prices of some finite number of goods in the third. The first system in the table, fluid flow, differs in having a continuum of continuous valued variables, the velocities $v(x)$, while the fourth, iterated social choice, has a finite number of discrete valued variables. In Example 1, these are the three pie allocations Mom proposes.

Our goal is to bridge the gap between this situation and the one exemplified by market economics and population dynamics—by making the natural generalization to processes where a continuum of allocations is allowed [5]. To varying degrees, this is a good model for social systems such as wealth/income distribution in populations, natural resource exploitation, market share for competing corporations, satellite bandwidth division among many users, and CPU time usage by multiple software agents running simultaneously. Specifically, we want to extend the techniques developed to quantify the complexity of iterated social choice [4] to continuous variables. We do so in Section 3, after reviewing the analysis of the discrete problem in Section 2. In Section 4 we consider a natural modification of the allocation process and then conclude with a discussion in Section 5.

2. Iterated discrete choice

The general setting for the discrete choice problem of Example 1 is a finite set of alternatives A and a finite number of agents, each of whom has a *preference order* on A. The usual model for the preference order of a rational agent is a relation, denoted $\geq$, which is *complete* ($a, b \in A \Rightarrow a \geq b$ or $b \geq a$) and *transitive* ($a, b, c \in A$ and $a \geq b$, $b \geq c \Rightarrow a \geq c$) [2]. We formalize voting or aggregation by maps f from preference profiles p (a list of agent preference orders) to directed graphs f_p. As in Figure 1, the vertices of f_p correspond to alternatives in A and a directed edge $a \leftarrow b$ in f_p indicates that for profile p the map f chooses alternative a over alternative b. We call f a *voting rule* if for all profiles p, f_p is *complete* ($a, b \in A \Rightarrow a \leftarrow b$ or $b \leftarrow a$ in f_p) and *unanimous* ($a, b \in A$ and $a \geq b$ in each preference order in $p \Rightarrow a \leftarrow b$ in f_p). Notice that in Figure 1 we have omitted the directed edges—present in every f_p—connecting each vertex to itself.

The directed graph f_p defines a *symbolic dynamical system*: Suppose the agents are presented with a sequence of alternatives. The results of successive pairwise votes between each new alternative and the current one form a sequence of symbols representing the chosen alternatives. The possible sequences are exactly the directed paths in f_p. These paths, together with the *shift map* (deletion of the first symbol of a sequence) form a dynamical system—a (*one-sided*) *subshift of finite type* [6]. To enumerate the paths we define the *transition matrix* F_p by $(F_p)_{ab} = 1$ if $a \leftarrow b$ in f_p and $(F_p)_{ab} = 0$ otherwise; then the number of paths of length N from b to a is $(F_p^N)_{ab}$. The *topological entropy* [7] of the symbolic dynamical system defined by f_p is

$$S[f_p] := \lim_{N\to\infty} \frac{1}{N} \log \operatorname{Tr} F_p^N = \log(\text{largest eigenvalue of } F_p). \tag{1}$$

In [4] we showed that the topological entropy is positive exactly when there is a Condorcet cycle and observed that this makes the dynamical system chaotic [8].

In Example 1 there is a preference cycle, the transition matrix is

$$F_p = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix},$$

and $S[f_p] = 2$. We can observe that the system is chaotic since, for example, arbitrarily close paths diverge like 2^t under t iterations of the shift dynamics. By contrast, suppose that in our example only Alice's preferences matter: since she prefers $a_1 > a_3 > a_2$, the direction on the edge between a_1 and a_2 in Figure 1 would be reversed. For this *dictatorial* voting rule f' we have an acyclic f_p', the transition matrix is

$$F_p' = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix},$$

and $S[F_p'] = 0$. This system is *not* chaotic: the set of paths converging to a_1 for large t has measure 1; neither is it complex: the whole system can be described by Alice's preference order—there is no difference between efficient models at the local (individual) and global (group) scales. The original system, however, is efficiently described at the global scale by f_p—a directed graph containing a nontrivial cycle, which is qualitatively different than an individual preference order.

These examples illustrate that the fundamental distinction between iterated discrete choice systems is *topological*—the presence or absence of preference cycles. Their existence was observed by Condorcet [1]; Arrow analyzed conditions under which they are inevitable [2]; Chichilnisky recognized their topological importance [9]; and we have noted that the topological entropy quantifies the amount of cyclicity, distinguishing simple from complex discrete choice systems [3], and suggested that it be used to quantify complexity in this context [4]. Now we will extend this approach to a situation with a continuum of alternatives—continuous allocation processes.

3. Continuous allocation processes

EXAMPLE 2. Suppose Mom is able to cut the pie into any three portions x_1, x_2 and x_3 for Alice, Bob and Charlie, respectively. Since there is only one pie, each allocation is described by a vector $x = x_1\hat{e}_1 + x_2\hat{e}_2 + x_3\hat{e}_3 \in \mathbb{R}^3$ with $x_1 + x_2 + x_3 = 1$ and $x_i \geq 0$. The x_i are barycentric coordinates on the *allocation simplex*—the two dimensional simplex bounded by the triangle with vertices at $\{\hat{e}_i\}$ ($\hat{e}_1 := (1,0,0)$, *etc.*). We supposed that each child prefers more pie to less, so the i^{th} child prefers allocation $y = (y_1, y_2, y_3)$ to $x = (x_1, x_2, x_3)$ iff $y_i > x_i$. This defines a profile for three agents on the space of allocations. Let us suppose, as in Example 1, that the three preference orders in this profile are aggregated by majority rule, *i.e.*, the group prefers allocation y to x iff

$$\sum_i \operatorname{sign}(y_i - x_i) \geq 0, \tag{2}$$

in which case we write $y \leftarrow x$.* The consequences of this voting rule are captured by the lightcone-like diagram shown in Figure 2: The point labelled x in the allocation simplex is preferred under majority rule to all points in the shaded regions, while all points in the unshaded regions, *e.g.*, a_1, are majority preferred to x. Possible sequences of outcomes in an iterated voting procedure then, consist of points each of which lies in 'the future lightcone'—the unshaded region—of its predecessor. Unlike the (usual) situation in Lorentzian geometry, however, cycles are prevalent: *e.g.*, the $a_1 \to a_2 \to a_3 \to a_1$ cycle of Example 1 exists also in this continuous generalization.

Our goal is to measure the complexity of this system by extending the topological entropy analysis we described for discrete choice systems [4] in Section 3. One might attempt to do so by *discretizing* Example 2 to some finite number V of allocations—such as the vertices of the triangular lattice shown in Figure 2—and then taking the limit as $V \to \infty$. The voting rule (Eq. 2) defines a directed graph on these vertices and we can compute the topological entropy (Eq. 1) of the associated symbolic dynamical system. As we refine the discretization to more vertices, however, the largest eigenvalue of the transition matrix, $\Lambda_V \sim V$ as $V \to \infty$, so to get a continuum limit we must rescale to Λ_V/V. The *relative entropy* [11] is the logarithm of the rescaled largest eigenvalue, *i.e.*,

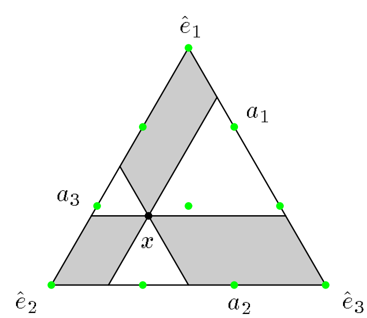

Figure 2. Aggregated preferences on the allocation simplex: unshaded points are majority preferred to x. The triangular lattice including alternatives a_1, a_2 and a_3 is a discrete approximation to the continuous allocation simplex.

$$S := \lim_{V\to\infty} \log \Lambda_V - \log V. \tag{3}$$

* With this voting rule the situation is equivalent to the general three-person zero-sum game considered originally by von Neumann and Morgenstern [10].

Since $\Lambda_V \leq V$, the relative entropy is nonpositive, vanishing for the maximally cyclic case of complete directed graphs. The relative entropy, therefore, measures how far the system is from being *maximally* chaotic.

To compute the relative entropy, however, it is more convenient—as well as aesthetically pleasing—to work directly in the continuum, where a transition matrix becomes a transition *operator*, or Green's function. Here it is defined by

$$T(y,x) := \begin{cases} 1 & \text{if } y \leftarrow x; \\ 0 & \text{otherwise.} \end{cases}$$

Matrix multiplication is replaced by integration: y can be reached from x in *two* steps provided

$$T^2(y,x) := \int_\triangle T(y,z)T(z,x)\mathrm{d}z > 0,$$

and in N steps if

$$T^N(y,x) := \int_\triangle T^{N-1}(y,z)T(z,x)\mathrm{d}z > 0,$$

where $\triangle$ indicates that the variable z is integrated over the allocation simplex. Just as in the discrete situation, the 'number' of cyclic paths of length N is given by the trace of this operator, which is again defined by integration. Thus the relative entropy of Eq. 3 can equivalently be defined as:

$$S := \lim_{N\to\infty} \frac{1}{N} \log \int_\triangle T^N(x,x)\mathrm{d}x - \log \operatorname{area}(\triangle), \tag{4}$$

i.e., as the logarithm of the largest eigenvalue of the operator T, rescaled by the area of the allocation simplex.

The eigenvalue problem for such an integral operator is

$$\lambda f(y) = \int_\triangle T(y,x)f(x)\mathrm{d}x. \tag{5}$$

Clearly $f(x) \equiv 0$ solves this equation; the problem is to determine the *eigenvalues* λ_i (in our case we mostly care about the one, $\Lambda := \lambda_1$, with the largest norm) for which Eq. 5 has nontrivial solutions, the *eigenfunctions*. Leaving the calculational details for another paper [12], Figure 3 shows the eigenfunction of T with the largest norm eigenvalue, $\Lambda \approx 0.28494$. Notice that $\Lambda < \frac{1}{4}\sqrt{6} = \operatorname{area}(\triangle)$ so that the relative entropy (Eq. 4) $S := \log\Lambda - \log \operatorname{area}(\triangle) < 0$. That is, there is a lower density of Condorcet cycles than there would be if $T(y,x) \equiv 1$

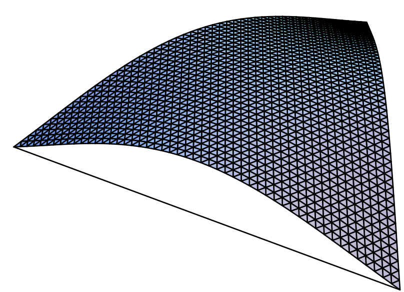

Figure 3. The approximate eigenfunction of T with largest norm eigenvalue $\Lambda \approx 0.28494$.

(in which case the largest norm eigenvalue solving Eq. 5 is $\frac{1}{4}\sqrt{6}$ and the corresponding eigenfunction is constant), but it is high enough that the relative entropy is finite.

As in the previous section, we can compare with a *simple* system: Suppose again that only Alice's preference matter so that there are no preference cycles. Then the transition operator is

$$T'(y,x) := \begin{cases} 1 & \text{if } y_1 \geq x_1; \\ 0 & \text{otherwise,} \end{cases}$$

and it is clear that the only nontrivial eigenfunction of this operator is concentrated at $\hat{e}_1$. Thus the eigenfunction is proportional to $\delta(\hat{e}_1 - x)$ and according to Eq. 5, the corresponding eigenvalue is 0. In this case the relative entropy is $-\infty$.

These examples demonstrate that the finiteness of the relative entropy (Eq. 3 or Eq. 4) for iterated continuous choice systems is the criterion corresponding to positivity of the entropy (Eq. 1) for iterated discrete choice systems. Each reflects the presence of topologically nontrivial paths in the space of alternatives* and identifies the system as complex.

4. Inequity

The eigenfunction of T shown in Figure 3 also reflects the presence of Condorcet cycles—by not being concentrated at a single point as is the eigenfunction of the transition operator T' modelling Alice's dictatorship. We may consider both the discrete and continuous choice systems probabilistically: at each timestep a new alternative y is presented at random—uniformly from all the alternatives. This probability distribution, weighted by $T(y,x)$, defines a transition probability from x to y. The eigenfunction is a projectively stationary distribution for this non-probability preserving process.

For many allocation processes, however, this is not the most realistic probabilistic model. In the case of wealth distribution, for example, it seems likely that a possible change in the allocation will be fairly small—just a perturbation of the *status quo* implemented by, say, a small change in the tax code. Rather than a uniform distribution for the new alternative, therefore, we might consider a Gaussian distribution centered at the *status quo*. The voting rule (Eq. 2) still determines which new alternatives are preferred. Figure 4 shows the first 400 steps of a path starting

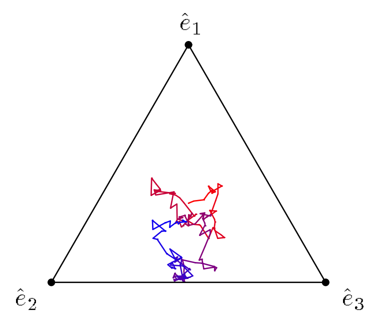

Figure 4. The first 400 steps of a path starting at the equal allocation $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and driven by a Gaussian random process with $\sigma = 0.03$.

* It is amusing to note that if we extend our model to continuous *time* choices, there are continuous periodic paths in the allocation simplex, as well as continuous paths from any point to any point other than the vertices $\hat{e}_i$. The discrete time version of this statement was proved by Ward [5].

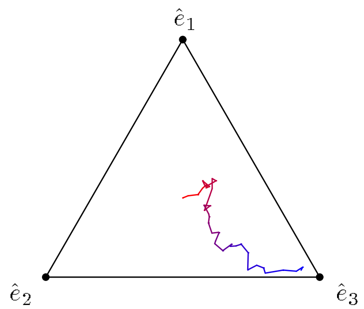

Figure 5. The first 100 steps for the inequitable rule (Eq. 6) driven by Gaussian fluctuations with $\sigma = 0.03$. There are cycles, but the path converges to $\hat{e}_3$.

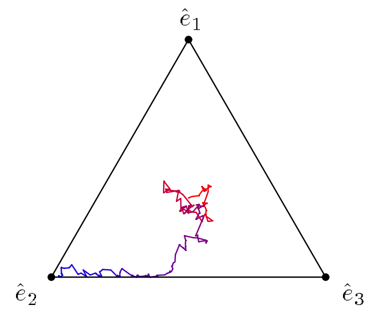

Figure 6. The first 400 steps for the same process driven by a narrower Gaussian ($\sigma = 0.02$). The path takes x_1 near zero and then converges to $\hat{e}_2$.

at the equal allocation $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and generated by a Gaussian with width $\sigma = 0.03$. Notice the cycles in the path, and the apparently larger probability for allocations near the center of the allocation simplex, both phenomena we derived in the previous section.

We must also remark that majority rule (Eq. 2) may be an unrealistic idealization for some allocation processes. In the case of wealth distribution, for example, it seems likely that the wealthier agents will have a greater say in the decision about a new allocation. Similarly, changes in market share may be more heavily influenced by corporations which dominate the market. To model unequal influences we need only weight the corresponding terms in Eq. 2, defining the aggregate preference $y \leftarrow x$ iff

$$\sum_i x_i \mathrm{sign}(y_i - x_i) \geq 0. \tag{6}$$

This voting rule weights the preference of each agent proportionaly to his/her fraction in the current allocation. Figures 5 and 6 illustrate the consequences of this inequitable voting rule with sample paths generated by Gaussian distributions. Each path starts at the equal allocation $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Figure 5 shows the first 100 steps with a Gaussian of width $\sigma = 0.03$, just as in Figure 4. Notice that while there are still cycles, the path converges rapidly to $\hat{e}_3$. Figure 6 shows the first 400 steps with a narrower Gaussian, $\sigma = 0.02$. In this case the allocation process takes longer to leave the center of the allocation simplex, and squeezes agent 1's fraction down near 0 before converging to $\hat{e}_2$.*

These examples illustrate that although the voting rule (Eq. 6) is symmetric (under permutation of the agents [10]), the fact that it is inequitable leads to *symmetry breaking* [13]: Almost all paths converge to one of the $\hat{e}_i$, but to *which* is a consequence of the Gaussian random process—deviations from the equal distribution are amplified and eventually 'frozen in' by the inequity of the voting rule (Eq. 6). So complex allocation processes can include the venerable economics phenomenon of increasing returns [14] and demonstrate the path-dependent consequences [15].

* Log normal (*i.e.*, Gaussian for the log of a multiplicative factor) distributions for the x_i, rescaled to keep the total 1, give similar—although not identical—results.

5. Discussion

We have generalized the entropy measure of complexity for iterated discrete choice systems to the economics setting of continuous allocation processes. The most complex system we have considered is the majority rule allocation process; in Section 3 we showed that the relative entropy can be computed precisely in this case. Furthermore, the density of Condorcet cycles makes the equal allocation $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ only the most probable; even with this equitable voting rule the expected values satisfy

$$\langle \text{max. fraction} \rangle > \frac{1}{3} > \langle \text{min. fraction} \rangle,$$

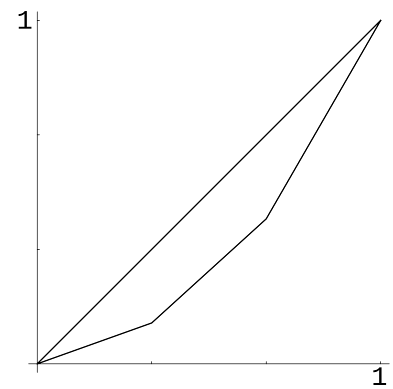

Figure 7. The Lorenz curve for the expected allocation of the eigenfunction shown in Figure 3 lies below the Lorenz curve for an equal distribution (the diagonal line). Fraction of the total is graphed as a function of 'poorest' population fraction.

implying inequality in the distribution. In Figure 7 we illustrate this inequality with a Lorenz curve [16] which plots the cumulative allocated fraction as a function of the 'poorest' fraction of the population. For the distribution given by the eigenfuntion in Figure 3, the Lorenz curve lies below the diagonal Lorenz curve for the equal allocation.

In Section 4 we considered allocation processes driven away from an initial equal allocation by Gaussian fluctuations. The expected inequality of these random processes increases in each case, stabilizing at some intermediate value for majority rule (Eq. 2), but converging to the maximum for the inequitable rule (Eq. 6). The latter situation exemplifies the evolution of a complex system into a simple one!

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