DECOHERENCE IN THE DIRAC EQUATION

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ABSTRACT

A Dirac particle is represented by a unitarily evolving state vector in a Hilbert space which factors as $H_{\text{spin}} \otimes H_{\text{position}}$. Motivated by the similarity to simple models of decoherence consisting of a two state system coupled to an environment, we investigate the occurence of decoherence in the Dirac equation upon tracing over position. We conclude that the physics of this mathematically exact model for decoherence is closely related to *Zitterbewegung*.

Decoherence—the process by which a quantum system initially in a pure state evolves into a mixed state as it interacts with its environment—has been studied both in toy models designed to elucidate aspects of quantum measurement [1–4] and in more complex ones modelling realistic physical situations [3,5–7]. In the purely quantum mechanical models the Hilbert space factors as a tensor product $H = H_{\text{sys}} \otimes H_{\text{env}}$ of Hilbert spaces describing the degrees of freedom of the system and environment, respectively. From the complete state $|\Psi\rangle \in H$ one forms the density operator $\rho := |\Psi\rangle \otimes \langle\Psi|$, which is traced over H_{env} to give the reduced density operator $\tilde{\rho} := \text{Tr}_{\text{env}}\rho$. Physically, the trace corresponds to the absence of measurements on the environment; $\tilde{\rho}$ is the average over all possible states thereof. Generically, $\tilde{\rho}$ is not of the form $|\Psi_{\text{sys}}\rangle \otimes \langle\Psi_{\text{sys}}|$ for any $|\Psi_{\text{sys}}\rangle \in H_{\text{sys}}$; it is rather a linear combination $\sum p_i |\psi_i\rangle \otimes \langle\psi_i|$, where $|\psi_i\rangle \in H_{\text{sys}}$ and the coefficients p_i are positive and sum to 1. That is, $\tilde{\rho}$ describes a mixed rather than a pure state, the degree of mixing being measured by the entropy $S := -\text{Tr}\tilde{\rho}\log\tilde{\rho}$.

The simplest decoherence models considered typically have a two dimensional H_{sys} coupled to an environment whose Hilbert space is also two [1–3] or at most finite [2,4] dimensional, although more realistic (infinite dimensional) environments have been used to investigate the absence of chiral superpositions in molecules [3], for example. Now, the Dirac equation describes the unitary evolution of a spin-$\frac{1}{2}$ particle—the Hilbert space factors as $H = H_{\text{spin}} \otimes H_{\text{position}}$—and H_{spin} is finite dimensional. In the spinor chain path integral formulation of the Dirac equation [8,9], to each direction in space is associated a spin projection, and over a time interval δ a projection operator is understood to propagate the particle a distance δ in the associated direction. A single path contributing to a transition amplitude from x to x' consists of a sequence of spin projections and chiralities such that the sum of the translations associated with the spin projections, each multiplied by the corresponding chirality, is proportional to the difference in position $x' - x$, where the proportionality constant depends on the spatial dimensionality. The amplitude of such a path is proportional to $(i\delta m)^r$ (in units with $\hbar \equiv 1 \equiv c$), where m is the mass of the particle and r is the number of chirality reversals.

From the point of view of decoherence models, this path integral describes exactly how the spin degrees of freedom are 'recorded' by the position of the particle. That is, H_{position} can be thought of as the Hilbert space of an environment measuring a quantum system with Hilbert space H_{spin}. Thus the Dirac equation can be analyzed for decoherence *via* reduced density matrices. In $1 + 1$ dimensions* the particle wave function is a two component spinor field $\psi(x,t) = \big(\psi_{-1}(x,t), \psi_{+1}(x,t)\big)$ in the chiral basis, so the density function is $\rho(x, \alpha, x', \alpha', t) = \psi_\alpha(x,t)\overline{\psi_{\alpha'}(x',t)}$, for $\alpha, \alpha' \in \{\pm 1\}$. Tracing over the environment is here an integration over position, so the reduced density matrix has four components

$$\tilde{\rho}_{\alpha\alpha'}(t) = \int \psi_\alpha(x,t)\overline{\psi_{\alpha'}(x,t)}\,dx. \tag{1}$$

In this note we examine (1) for decoherence by computing, exactly, its entropy as a function of time.

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* Since there is no spin—only chirality—in $1+1$ dimensions, this will simplify the subsequent discussion with no conceptual loss and allow direct comparison with decoherence of other two state systems [1–4].

Using the path integral formulation of the Dirac equation it is straightforward to compute the propagator $K_{\alpha_2\alpha_1}(x_2, t_2, x_1, t_1)$ satisfying

$$\psi_{\alpha_2}(x_2, t_2) = \int_{x_2-\Delta t}^{x_2+\Delta t} K_{\alpha_2\alpha_1}(x_2, t_2, x_1, t_1)\psi_{\alpha_1}(x_1, t_1)\, dx_1, \tag{2}$$

where $\Delta t := t_2 - t_1$. The result is [8,9,10]:

$$K_{\alpha_2\alpha_1}(x_2, t_2, x_1, t_1) = \begin{cases} \delta(\Delta t - \alpha_1\Delta x) - (\Delta t + \alpha_1\Delta x)mJ_1(m\tau)/2\tau & \text{if } \alpha_2 = \alpha_1; \\ imJ_0(m\tau)/2 & \text{if } \alpha_2 \neq \alpha_1, \end{cases} \tag{3}$$

where $\Delta x := x_2 - x_1$, $\tau := \sqrt{(\Delta t)^2 - (\Delta x)^2}$ and J_i denotes the ith order Bessel function of the first kind. Using (2) and (3) in (1) gives an exact formula for the reduced density matrix as a function of time in terms of the initial state of the particle.

Decoherence is usually studied for an initial state which is a tensor product pure state $|\Psi_{\text{sys}}\rangle \otimes |\Psi_{\text{env}}\rangle$ [1–7], so consider the case $\psi(x, 0) = f(x)\binom{1}{1}$, where $f(x)$ is proportional to $e^{-x^2/2}$ and is normalized so that $\int \psi^\dagger(x, 0)\psi(x, 0)\, dx = 1$. That is, the particle is initially in an equal superposition of negative and positive chiralities and is localized near $x = 0$. Figure 1 shows the entropy $S(t)$ of the reduced density matrix (1) for the range of masses $m \in \{0, 1, 2\}$. Near $t = 0$ the rate of entropy gain increases as the mass decreases: the top curve is for $m = 0$ while the bottom one is for $m = 2$. Thus, as could have been anticipated from the amplitudes associated to paths in the spinor chain path integral, m^{-1} controls the strength of the system/environment interaction: when m^{-1} is larger, decoherence occurs more rapidly.
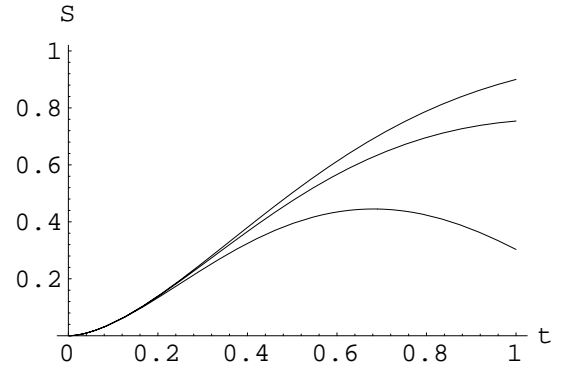

Figure 1. Entropy as a function of time for the reduced density matrix (1) with the particle in an initial equal superposition of chiralities and localized in a Gaussian around $x = 0$. From top to bottom, the curves are for increasing mass.

To understand this decoherence, we begin by considering $m = 0$. In this case the Bessel function terms in (3) vanish so the effect of the propagator is simply to translate the initial chirality position amplitudes along the lightcone: $\psi(x, t) = f(x + t)\binom{1}{0} + f(x - t)\binom{0}{1}$. For the initial position Gaussian we can compute the reduced density matrix (1) exactly to get:

$$\tilde{\rho}(t) = \frac{1}{2}\begin{pmatrix} 1 & e^{-t^2} \\ e^{-t^2} & 1 \end{pmatrix}. \tag{4}$$

The diagonal terms in (4) are constant and equal, while the off-diagonal terms go rapidly to 0; thus the entropy goes rapidly to 1 as indicated by Figure 1.

Next, we consider $m = 1$. Just as in the massless case, if we were to plot the position distributions for each chirality at $t = 0$, *i.e.*, $|\psi_{-1}(x, 0)|^2$ and $|\psi_{+1}(x, 0)|^2$, they would
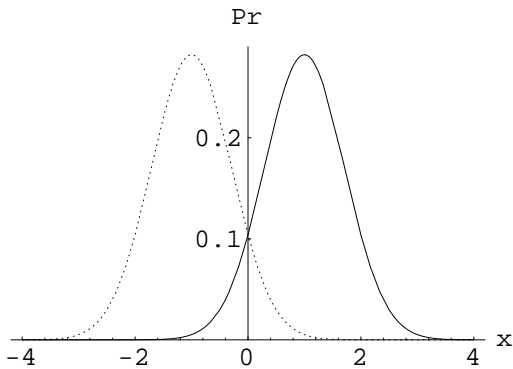
Figure 2. The $t = 1$ chirality position distributions for $m = 0$. The distribution with the peak at the left/right is for negative/positive chirality; each remains a Gaussian.
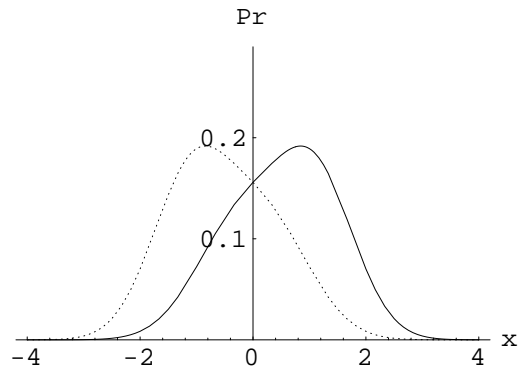
Figure 3. The same chirality position distributions shown in Figure 2 but for the case $m = 1$. Dispersion distorts the initial Gaussian and reduces the separation of the distributions.

be two coincident Gaussians centered at $x = 0$; there is no initial correlation between position and chirality for a tensor product state. Figures 2 and 3 show the chirality position distributions at $t = 1$ for $m = 0$ and $m = 1$, respectively. In each case the distributions separate: a particle with negative/positive chirality tends to move to the left/right. Now chirality is correlated with position, more strongly for $m = 0$ than for $m = 1$, and it is this correlation which is measured by the entropy upon tracing over position. For $m = 1$ dispersion distorts the initial Gaussian so that there is a weaker correlation between chirality and position than for $m = 0$, and thus lower entropy.

Since the spin system is coupled to an infinite dimensional position Hilbert space environment, not a small finite dimensional Hilbert space with a short Poincaré recurrence time [2–4], one might have expected even this weaker chirality/position correlation to continue to build and the entropy to increase asymptotically to 1 just as it does for $m = 0$. This expectation is contradicted, however, by the $m = 2$ curve in Figure 1. In this case the entropy reaches a (local) maximum and then begins to decrease. In fact, the oscillatory behavior beginning here characterizes all of the massive entropy curves.

To explain how the entropy can decrease, despite the infinite dimensional position Hilbert space, Figure 4 shows the $m = 1$ entropy curve over a longer time interval $0 \leq t \leq 2$. The inset graphs show the same chirality position distributions plotted in Figures 2 and 3, for times $t \in \{0.5, 1, 1.5, 2\}$. The slowing and then reversal of the entropy increase is due to the decreasing portion of the distributions in the fastest moving (outermost) peaks and the resulting increasing portion of each distribution which overlaps the other.* That is, dispersion acts initially to increase (by separating the distributions) and subsequently to decrease (by broadening/distorting the distributions) entropy; then the cycle repeats. For smaller masses not only does the initial increase occur faster, but the entropy also reaches greater values before starting to decrease.

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   * Spacetime plots of quantum lattice gas automaton simulations of the Dirac equation showing exactly this behavior (although with periodic spatial coordinate) can be found in [10].
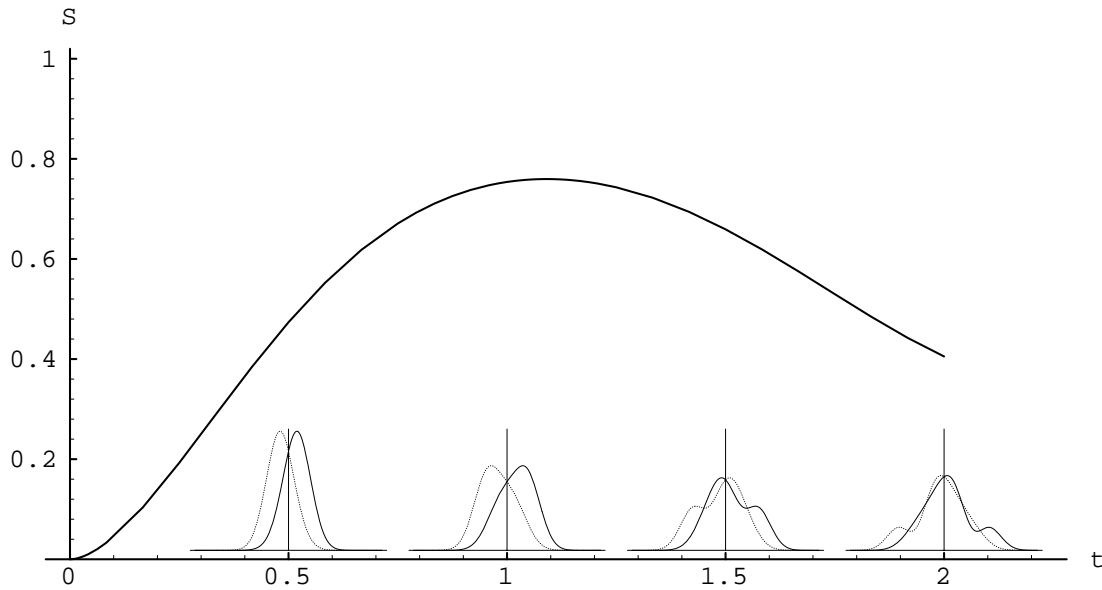
Figure 4. Entropy plotted as a function of time for an initial position Gaussian with $m = 1$. The inset graphs show the chirality position distributions at half integer times.

The results shown in Figures 1–4 have been obtained for quite special initial conditions: in each case an equal superposition of negative and positive chiralities. While this choice demonstrates decoherence in the Dirac equation particularly clearly, any spinor tensored with an initial position Gaussian will decohere. Figure 5 shows the entropy curve over $0 \leq t \leq 1$ for initial condition $\psi(x,0) \propto e^{-x^2/2}\binom{0}{1}$ with $m = 1$, while Figure 6 shows the chirality position distributions at $t = 0.5$. The larger distribution is $|\psi_{+1}(x,0.5)|^2$; as expected, since the initial chirality is purely positive, the chirality is still more likely to be positive than negative at $t = 0.5$. Nevertheless, there is some probability that the particle
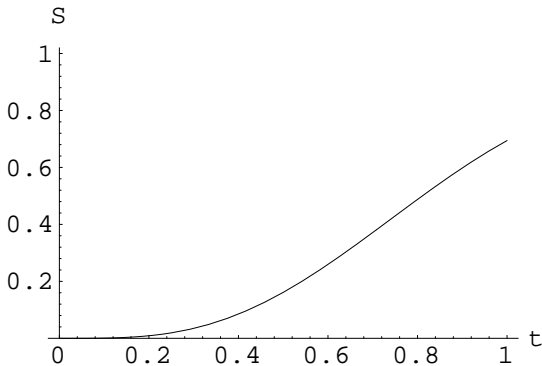
Figure 5. Entropy as a function of time for the chiral initial condition localized by a position Gaussian and for $m = 1$. Initial entropy gain is slower than in the corresponding parity invariant case shown as the middle curve in Figure 1.
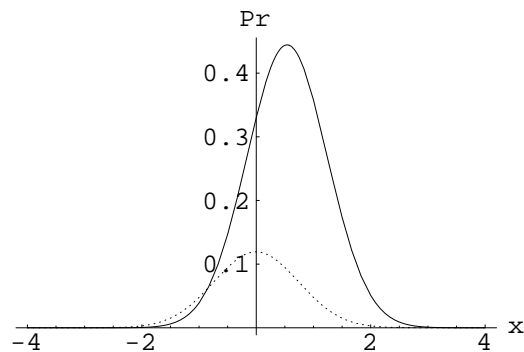
Figure 6. The distributions $|\psi_{\pm 1}(x,0.5)|^2$ for the same initial condition as in Figure 5. The larger distribution is for positive chirality; the smaller, for negative. There is a small chirality/position correlation.

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has negative chirality, so chirality/position correlation has developed. Just as in the parity invariant cases, only more slowly, this correlation causes the entropy increase shown in Figure 5.

By superposition in H_{sys}, therefore, any tensor product initial condition of the form $\psi(x, 0) \propto e^{-x^2/2}(a, b)$ decoheres. This result, and its familiarity in the context of other models of decoherence [1–7], might lead us to believe that *any* tensor product state, initially localized in position, decoheres—and to conclude that the spin of a Dirac particle evolves, absent position measurement, stochastically rather than purely quantum mechanically. This, of course, is *not* true, as we shall see by working in the stationary state basis.

The stationary states of the Dirac equation are the plane waves $\psi^{(k,\epsilon)}(x) = e^{ikx}u(k, \epsilon)$, where $k \in \mathbb{R}$, $\epsilon \in \{\pm 1\}$, and the fixed spinors $u(k, \epsilon) \in \mathbb{C}^2$ are normalized and orthogonal for $\epsilon = 1$ and -1. While it is not localized in space, of course, suppose we take a plane wave $\psi^{(k,\epsilon)}$ as our initial tensor product state. Since this stationary state evolves by multiplication by the phase $e^{-i\epsilon\omega t}$, where the energy $\epsilon\omega$ satisfies the dispersion relation $\omega^2 = k^2 + m^2$, the density operator is constant:

$$\rho(x, x', t) = e^{ik(x-x')}u(k, \epsilon) \otimes u^\dagger(k, \epsilon) =: e^{ik(x-x')}\tilde{\rho}(t),$$

as is the reduced density matrix. Thus stationary states do *not* decohere, a familiar result in decoherence models with tensor product eigenstates of the Hamiltonian [1,3,4].

Any initial state can be expanded as a superposition of plane waves, *i.e.*,

$$\psi(x, 0) = \sum_\epsilon \int dk\, \hat{\psi}(k, \epsilon)e^{ikx}u(k, \epsilon).$$

In this form the time evolution is transparent:

$$\psi(x, t) = \sum_\epsilon \int dk\, \hat{\psi}(k, \epsilon)e^{ikx-i\epsilon\omega t}u(k, \epsilon),$$

so it is easy to write down the time dependent density operator explicitly:

$$\rho(x, x', t) = \sum_{\epsilon,\epsilon'} \int dk dk'\, \hat{\psi}(k, \epsilon)e^{ikx-i\epsilon\omega t}u(k, \epsilon) \otimes u^\dagger(k', \epsilon')e^{-ik'x'+i\epsilon'\omega't}\overline{\hat{\psi}(k', \epsilon')}. \qquad (5)$$

To obtain the time dependent reduced density matrix we need only set $x' \equiv x$ in (5) and integrate over x. The integral of $e^{i(k-k')x}$ gives a delta function of $k-k'$, so we can evaluate the integral over k' to get

$$\tilde{\rho}(t) = \sum_{\epsilon,\epsilon'} \int dk\, \hat{\psi}(k, \epsilon)e^{-i\epsilon\omega t}u(k, \epsilon) \otimes u^\dagger(k, \epsilon')e^{i\epsilon'\omega t}\overline{\hat{\psi}(k, \epsilon')}. \qquad (6)$$

The only nontrivial time dependence of $\tilde{\rho}(t)$ derives from terms in (6) with a factor of the form $e^{-i(\epsilon-\epsilon')\omega t}$ for $\epsilon' \neq \epsilon$. Thus an initial state (whether a tensor product or not) decoheres only if it has nonzero amplitudes $\hat{\psi}(k, \epsilon)$ and $\hat{\psi}(k, -\epsilon)$ for some $k \in \mathbb{R}$. That is, decoherence depends on the presence of positive and negative energy modes of the same momentum. The initial states whose decoherence we exhibited in Figures 1–5 are superpositions of such modes; in contrast, a wave packet constructed from only positive energy plane waves, would disperse, of course, but not decohere.

We conclude by remarking that while the Dirac equation provides a mathematically exact model of decoherence, this decoherence is another aspect of the physical difficulties with the one particle interpretation of the Dirac equation. The particle decoheres and behaves stochastically rather than quantum mechanically exactly to the extent that it is not identified as an electron or positron, for example. The same interference between positive and negative energy states leads to *Zitterbewegung* [11]. That the entropy oscillates, rather than increasing permanently as it does in other decoherence models with infinite dimensional environment Hilbert spaces [5–7], is a consequence of the same oscillating factors $e^{-2i\epsilon\omega t}$ in (6) that cause *Zitterbewegung*. To revise FitzGerald's rendering of *The Rubáiyát* of Omar Khayyám [12]:

> The Moving Finger writes; and, having writ,
> *Zitterbewegt sich.*

Acknowledgements

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