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THE COMPLEXITY OF VOTING

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ABSTRACT

Voting, or more generally, aggregating preferences, makes systems complex. Consequently, their dynamics can be chaotic and hence difficult to predict or control. These statements are equally true for artificial multi-agent systems as for real political systems, counterindicating unreflexive use of such tools by policy makers. Similar analysis applies to systems with a continuum of alternatives, like election campaigns, and explains the ambiguity of candidates' platforms.

Key Words: decision theory, chaos, Arrow's theorem, multi-agent simulations, agent-based modelling, political campaigns, platform ambiguity.

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1. Overview

This talk is not about the difficulty of deciding how to vote—although I will have something to say about that at the end—but rather about the sense in which voting, or more generally, aggregating preferences, creates complex systems ... and some consequences of this fact. Furthermore, I want to explain several reasons policy makers should care: Underlying their tasks is the basic problem of understanding (and influencing) group decision making. Advances in such understanding have implications for real social systems and for multi-agent tools, as well as for interpreting simulations of either. The approach I take is motivated by a lesson from physics: for many situations minimal models abstracting only crucial features provide useful insights into the behavior of real systems.

2. Complex systems

For our purposes a complex system is one composed of multiple subsystems interacting in such a way that efficient descriptions differ at different scales [1]. Larger scale properties which differ from smaller scale ones are often described as 'emergent' [2]. Let's consider two examples:

(1) A barbell with $90lbs$ of weight added: Suppose the $45lbs$ on each side consist of nine $5lb$ weights. Efficient description of a subsystem—$5lbs$ of weight—is of the same kind as of the whole—$45lbs$ of weight; only the number changes. This is a *simple* system.

(2) A hurricane: At a small scale it is composed of the molecules of the air, which are efficiently described by their positions and momenta. But at a (much) larger scale the hurricane is a vortex, which is efficiently described by the latitude and longitude of its center, and by its angular momentum. These descriptions are of different kinds, so this is a *complex* system.

Of course, sometimes winds blow in a single direction even at very large scales so that larger and larger volumes of air are efficiently described by position and momentum. Conversely, there are actually differences between loading a barbell with different sets of weights totalling the same amount—the bar vibrates differently and this matters when a lifter is at the limit of his/her strength. In fact, all real (classical) systems are complex, although they may be simple in special circumstances.

3. Social systems

Policy makers are concerned with social systems—political, economic, military, *etc.* These consist of multiple people with individual preferences which are aggregated by institutional mechanisms—voting, markets, and command, respectively. Efficient description changes at the scales of aggregation/decision making; thus these are complex systems. Voting exemplifies this phenomenon, which was noticed over 200 years ago by Condorcet [3].

He considered a situation with 60 voters, each having consistent preferences among three alternatives, a, b and c, as listed in Table 1:

	$a > b$	$b > c$	$c > a$
23 prefer $a > b > c$	23	23	
17 prefer $b > c > a$		17	17
2 prefer $b > a > c$		2	
10 prefer $c > a > b$	10		10
8 prefer $c > b > a$			8
majorities:	33	42	35

Table 1. Condorcet's example of a preference cycle.

Note that if the voters are asked to choose between a and b a majority prefer a, between b and c a majority prefer b, but between a and c the majority does not prefer a, but rather c. We can summarize the situation by the directed graph on the left in Figure 1, where the vertices represent alternatives and the direction of edges indicates majority preference. Compare this with the directed graph on the right in Figure 1 which describes the consistent preferences $c > a > b$. These graphs differ *topologically*: the former contains a *cycle* (like a hurricane) while the latter does not. Thus efficient description at the group level differs from the individual level and the system is complex.

Figure 1. Directed graphs representing the aggregated choices of two different groups. On the left, the group preferences contain a cycle; on the right they do not.

4. Dynamics

To understand the consequences of this complexity, consider the following dynamics: Suppose the voters are successively presented with a choice between the *status quo* and a new alternative. If the group preferences contain the cycle shown on the left in Figure 1, a typical sequence of choices is *bbaccbaa*.... In contrast, the acyclic preferences on the right lead to sequences of the form *bbaccccc*.... In the latter case, once alternative c is chosen, the sequence becomes constant, but in the former case, the group never settles on a single alternative.

The difference between these two situations can be quantified [4]: To a directed graph associate a transition matrix T with entries defined by

$$T_{ab} = \begin{cases} 1 & \text{if } a \leftarrow b; \\ 0 & \text{otherwise.} \end{cases}$$

Labelling the rows and columns a, b, c, in that order, the transition matrices for the directed graphs shown in Figure 1 are:

$$T_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad T_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix},$$

respectively. The number of admissible sequences of N choices is $\mathrm{Tr}\, T^N$, where Tr denotes the sum of the diagonal entries. The *topological entropy* normalizes this quantity:

$$S[T] := \lim_{N\to\infty} \frac{1}{N} \log \mathrm{Tr}\, T^N = \log(\text{largest eigenvalue of } T).$$

Thad Brown and I have show that this entropy is positive exactly when there are preference cycles and that in this case the dynamics of the system is chaotic [4]. Recall that the technical meaning of *chaotic* is that the system displays [5]:

(1) topological transitivity;
(2) a dense set of periodic orbits;
(3) sensitive dependence on initial conditions.

Concentrating on the last of these properties, notice that $S[T_1] = 1 > 0$. Even when sequences of choices admissible with the cyclic preferences in Figure 1 start out the same, once a difference occurs the sequences diverge: *e.g.*, *bbaccbaa*..., *bbaccbbb*..., *bbacbaac*.... Contrast this with the acyclic situation in Figure 1 for which $S[T_2] = 0$ and initial differences are erased once the sequence settles down to alternative c: *e.g.*, *bbaccccc*..., *bbaaaacc*..., *bacccccc*....

5. Real systems

To see how this plays out in real systems, let's consider two examples:

(1) Postwar Italy. Italy is notorious for the frequency with which it changes govenments, having averaged more than one government per year since the end of World War II. Figure 2 crudely illustrates the history [6]. Since Italy is a parliamentary democracy, the ruling government is often a coalition of several parties. The height of each rectangle in Figure 2 represents the number of participating

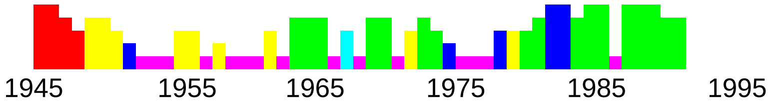

Figure 2. A crude representation of the governments in post-war Italy: *Democrazia Cristiana* (purple); *Partito di Rifondazione Comunista* (red); *Partito Socialista Lavoratori Italiano*, *Partito Socialista Democratico Italiano* (yellow); *Partito Republicana Italiano* (navy); *Partito Socialista Italiano* (green); *Partito Socialista Unificato* (light blue).

parties; since the Christian Democrats are the dominant party, the color codes the secondary party in the ruling coalition. The sequence of choices is consistent with cyclic rather than acyclic group preferences.

(2) U. S. Presidents 1960–1980. This provides a more familiar (to Americans) example which I've excerpted in Table 2. Not only does this history display the same alternation between parties (in this case, Democrat and Republican), but the details indicate that the larger social dynamics in which the presidential elections are embedded must also be sensitive to small perturbations: Had any of a number of things occurred differently, even the binary sequence of governing parties could easily have been different.

1960	Kennedy (D) beats Nixon by 118,000 votes out of 78,000,000
	Nixon loses campaign time to a car accident
	Daly delivers crucial precints—hence IL—for Kennedy
1963	Johnson becomes President after Kennedy assassination
1964	Johnson (D) re-elected
1968	Nixon (R) beats Humphrey by 500,000 votes out of 72,000,000
	near election, polls shifting towards Humphrey
	Wallace gets 9,900,000 votes in Democratic South
1972	Nixon (R) re-elected over McGovern
1973	Ford becomes VP after Agnew kickback conviction
1974	Ford (R) becomes President after Nixon's resignation
1974	Ford pardons Nixon
1976	Carter (D) beats Ford by 57 electoral votes out of 538

Table 2. Two decades of U. S. Presidents.

6. Arrow's theorem

A reasonable question at this point is: Could changing the decision procedure eliminate chaos? Each of voting rules I've discussed has been majority rule. Perhaps requiring supermajorities, or some more complicated rule, would eliminate the possibility of cyclic group preferences. That this is *impossible* is the content of Arrow's theorem [7]. More precisely, he showed that any voting rule which is

(1) preference preserving ($a > b$ for all voters implies $a > b$ for the group), and
(2) independent of irrelevant alternatives (the ranking of a and b for the group depends on the voters' rankings only of a and b),

is either dictatorial (*i.e.*, there is a single voter whose preferences dictate the group preferences) or there is a set of voter preferences so that the group preferences are cyclic ... and hence the dynamics is chaotic as described in §4.

The topological entropy defined in §4 can be used to quantify the in/stability of a voting rule, or more generally, an institution, by averaging over a set of possible preferences for the voting population. Conversely, averaging the entropy over a class of voting rules quantifies the in/stability of a specific voting population [4].

7. Policy implications

There are several immediate implication for policy makers:

Predicting outcomes of decisions into the future is exponentially difficult. For example, it's already hard to predict the winner of the next presidential election, and an answer to a question like, "Will the winner of the 2020 presidential election be a Democrat or a Republican?" can hardly be more than a guess.

Controlling outcomes is at least as difficult. This, of course, is something often attempted by the U. S. government ... usually with unforseen results. The upheavals in the Balkans in the 5 years since the Dayton Accords illustrate this, as does the post-1953 history of Iran and the last quarter century of events in Indonesia.

Analogous statements hold for any complex system. Important examples include financial markets, ecosystems and the global climate. All of these are domains in which future events are of great importance and over which some control is desirable or necessary. But the complexity of these systems makes small scale prediction and control intractable.

So what is a policy maker to do when faced with such a complex system? *Concentrate on system properties on scales at which they can be described, measured or manipulated, e.g., entropy/instability.* Detailed understanding at smaller scales requires exponentially increasing commitment of resources: data gathering and computational power, for example.

8. Modelling and simulation

The alternative to expending exponentially increasing effort is to create the kind of minimal models I mentioned in §1, and only expect them to provide certain kinds of macroscopic information. They should include all and only the interactions relevant to the scale of description desired [8]. This is not always straightforward and many realistic simulations (including some of those described at this workshop) include too much or too little detail.

The voting model I've discussed describes people by preference orders. This is a minimal model which seems very reasonable to physicists, but has been much debated by social scientists. There are contexts, however, in which such a model is unarguable. These include three modern problem solving techniques:

(1) multi-agent simulations, *e.g.*, Swarm [9];
(2) multi-agent (*e.g.*, ANT [10] or genetic [11]) algorithms;
(3) robots, *e.g.*, [12].

It is important for policy makers, who are increasingly often presented with one of these systems as a putative tool, to recognize that they have the same strengths and weaknesses as we have demonstrated in human group decision making: sensitivity to initial conditions, resilience to manipulation, and inescapable inefficiencies. Each of these deserves elaboration, but I do not have time in this talk to provide it.

9. Campaign physics

The last two sections contain the punchlines for this talk, but I promised at the beginning to say something about the difficulty of voting. This requires generalizing the system with discrete alternatives considered in §3–§6 to the common situation in which alternatives form a continuum [13], *e.g.*, 'left' and 'right' on a 'political spectrum'. If multiple issues are at stake, as in an election, they form a multi-dimensional 'policy space'. This is the arena for spatial voting models [14].

As a minimal model, suppose there are two dimensions (say domestic economy and civil rights) and voters who have 'ideal points' in this space, meaning each prefers a candidate who is closer to rather than further from his/her ideal point. Figure 3 shows three ideal points and a candidate's current position. Moving into any of the lens-shaped regions improves the candidate's ranking by two of the three voters. Assume a candidate always wants to increase his/her appeal to a majority. How will s/he campaign?

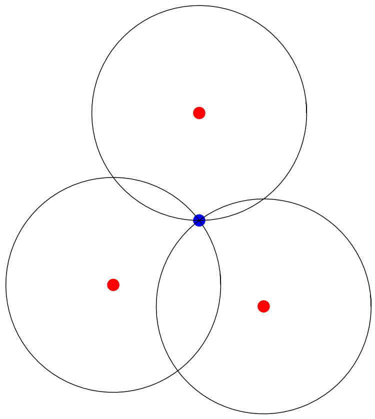

Figure 3. Three ideal points and a candidate's current position.

The analogue of the transition matrix defined in §4 is an operator

$$T(z,w) := \begin{cases} 1 & \text{if } z \leftarrow w; \\ 0 & \text{otherwise,} \end{cases}$$

for z, w points in the policy space. Now define the *relative entropy* to be

$$S[T] := \lim_{N\to\infty} \log \int T^N(z,z)\mathrm{d}z - \log(\text{area})$$

which is again log(largest eigenvalue of T), rescaled by the area of the policy space. Recall that the eigenvalues λ of T solve the integral equation

$$\lambda f(z) = \int T(z,w) f(w) \mathrm{d}w.$$

If an eigenfunction f is concentrated at a single point (the ideal point of a single voter, say), the largest eigenvalue $\Lambda = 0$ and the relative entropy is negative infinity. If there are cycles, however, f is not concentrated, $\Lambda > 0$, the relative entropy is finite, and the system is chaotic. The eigenfunction corresponding to Λ describes the relative frequency with which the candidate takes different positions in policy space. For the example of Figure 3, Figures 4 and 5 illustrate this frequency distribution.

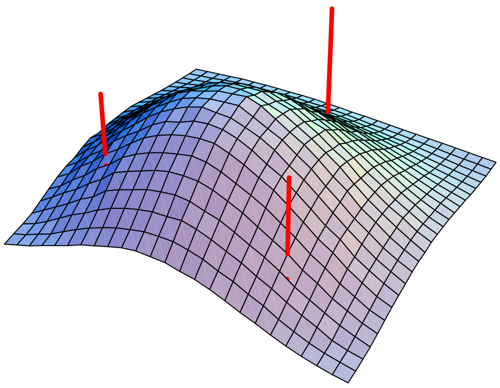

Figure 4. The distribution of positions a candidate takes for the exampe of Figure 3.

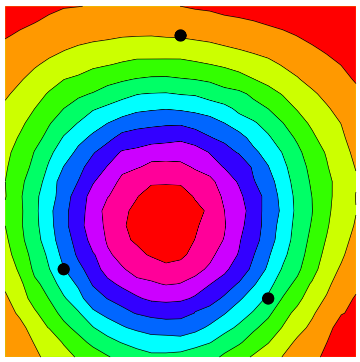

Figure 5. The 'contours of ambiguity'; another view of the same distribution.

From these figures we see that a candidate is motivated to campaign by making a sequence of position statements which end up distributed over policy space according to the eigenfunction f. Thus this minimal model provides an explanation for the fuzziness of candidates' platforms—and hence why it's so hard for us as voters to discern exactly for what they stand. In the 2000 U. S. presidential campaign, for example, Gore gave speaches in DC and in LA admitting of different interpretations for his position on movie violence, while Bush did the same thing with race relations at Bob Jones University and before the NAACP.

I'll conclude with a remark written after the election: This model applies equally well to a two candidate campaign in which the candidates iteratively try to outpoll each other. Absent any ideological constraints, their optimal distribution of positions is the same and voters have little to choose between. In fact, if both candidates run optimal campaigns—something which developments in polling and communications technologies are continually making easier to do—one should expect the voters to split equally between the candidates which is, of course, what happened this year ... so it seems unjustified to criticize either candidate for running a poor campaign. Unfortunately, this analysis does not tell us how to resolve the resulting tie!

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