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PARRONDO GAMES AS LATTICE GAS AUTOMATA

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ABSTRACT

Parrondo games are coin flipping games with the surprising property that alternating plays of two losing games can produce a winning game. We show that this phenomenon can be modelled by probabilistic lattice gas automata. Furthermore, motivated by the recent introduction of *quantum* coin flipping games, we show that quantum lattice gas automata provide an interesting definition for quantum Parrondo games.

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0. Introduction

The simplest quantum lattice gas automata (QLGA) provide discrete models for the $1+1$ dimensional Dirac equation [1,2] and the multiparticle Schrödinger equation [3]. More complicated QLGA can be constructed to model potentials [4], inhomogeneities and boundary conditions [5]. In this talk we motivate the introduction of a QLGA model from a completely new perspective—Parrondo games.

A Parrondo game is a sequence of plays of two simpler games, each of which involves flipping biased coins. In §1 we review the somewhat surprising result that even if each of the simpler games is a losing game, an alternating sequence of them can be a winning game [6,7]. Meyer has recently initiated the study of quantum game theory with an example of a coin flipping game, PQ PENNY FLIP [8]. This raises the natural question: Is there a quantum version of Parrondo games? Although the quantum Parrondo game we construct is not a two player game (as PQ PENNY FLIP is) it introduces a formalism for coherently iterated games which we expect to be useful in contexts involving one, two, or more players.

Parrondo invented the coin flipping game, however, to illustrate a physical phenomenon—Brownian ratchets [6,7]; in §2 we explain this connection in terms of a probabilistic discrete model—a random walk. This stochastic microscopic model captures the macroscopic irreversible behavior of ratcheting, but raises the concern that a microscopic quantum model which is exactly unitary may not be able to do so [9]. The more immediate difficulty is the absence of any unitary version of a random walk. To get a 'quantizable' model we must first generalize to a *correlated* random walk [10], or equivalently, a probabilistic LGA; we explain this in §3.

From here it is only a small step—actually an analytic continuation [11]—to a single particle QLGA. We review the unitary evolution rules in §4, emphasizing the inclusion of potentials which are necessary to model ratcheting. §5 contains the results of simulations which appear to illustrate quantum ratcheting, and which lead us to answer our motivating question by interpreting the single particle QLGA with appropriate potentials as a quantum Parrondo game. We conclude in §6 with a summary and some more physical observations.

1. Parrondo games

Consider games which involve flipping a coin: winning 1 when it lands head up and losing 1 when it lands tail up. Suppose there are three biased coins A, B_0, and B_1, with probabilities of landing head up of p_a, p_0, and p_1, respectively. Define game A to consist of repeatedly flipping coin A. For $p_a < \frac{1}{2}$, A is a losing game in the sense that if the initial stake is $x = 0$, after t plays the expected value of the payoff is $\langle x \rangle = t(2p_a - 1) < 0$. Even though one may win sometimes, in the long run one must expect to lose.

After each flip the payoff x changes by ± 1. Define game B to consist of repeatedly flipping coins B_0 and B_1: B_0 when $x \equiv 0 \pmod 3$ and B_1 otherwise. This defines a

Markov process on x (mod 3) with transition matrix

$$T_B = \begin{pmatrix} 0 & 1-p_1 & p_1 \\ p_0 & 0 & 1-p_1 \\ 1-p_0 & p_1 & 0 \end{pmatrix}. \tag{1}$$

The equilibrium state, *i.e.*, the eigenvector (v_0, v_1, v_2) of T_B with eigenvalue 1 (normalized by $v_i \geq 0$, $\sum v_i = 1$) determines the long time behavior of the game: for large t, the expected payoff is $\langle x \rangle = t[(2p_0 - 1)v_0 + (2p_1 - 1)(v_1 + v_2)]$. Thus B is a fair game iff the matrix

$$\begin{pmatrix} -1 & 1-p_1 & p_1 \\ p_0 & -1 & 1-p_1 \\ 2p_0-1 & 2p_1-1 & 2p_1-1 \end{pmatrix} \tag{2}$$

is singular, *i.e.*, iff

$$p_0 = \frac{1 - 2p_1 + p_1^2}{1 - 2p_1 + 2p_1^2}. \tag{3}$$

One specific solution to equation (3) is $(p_0, p_1) = (\frac{1}{10}, \frac{3}{4})$, but for a Parrondo game, B should be a losing game, which means choosing p_0 and p_1 such that LHS(3) $<$ RHS(3). Figure 1 plots $\langle x \rangle$ as a function of t for A and B games defined by $p_a = \frac{1}{2} - \epsilon$, $p_0 = \frac{1}{10} - \epsilon$, and $p_1 = \frac{3}{4} - \epsilon$, with $\epsilon = 0.005$. Each is clearly a losing game.

Now suppose we combine these games. More precisely, suppose they are played in the order $AABB$, repeatedly. Figure 1 plots the expected result of this game as well. Parrondo's 'paradoxical' observation is that this combination of two losing games is a winning game! To understand this phenomenon, rather than attempting to generalize the Markov process analysis of equations (1)–(3), let us go back to the physical system which motivated Parrondo.

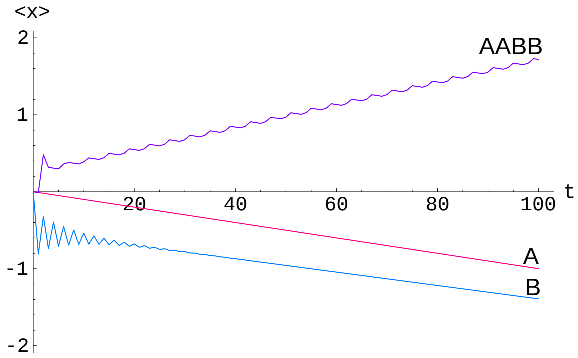

Figure 1. The expected payoffs for games B, A and $AABB$ as a function of number of plays t. Although A and B are losing games, the combination $AABB$ is a winning game.

2. Brownian ratchets

The payoff x for game A with $p_a = \frac{1}{2}$ executes an unbiased random walk on the integers, which is a discrete model for the diffusion equation in $1 + 1$ dimensions [12]:

$$\rho_t = D\rho_{xx}. \tag{4}$$

That is, the distribution $p(x, t) = \text{Prob}(\text{payoff} = x \text{ at time} = t)$ approximates $\rho(x, t)$ in (4) with $D = (\Delta x)^2/2\Delta t$. For $p_a \neq \frac{1}{2}$ the random walk is biased and is a discrete model for diffusion with linear advection [12]:

$$\rho_t + c\rho_x = D\rho_{xx}, \tag{5}$$

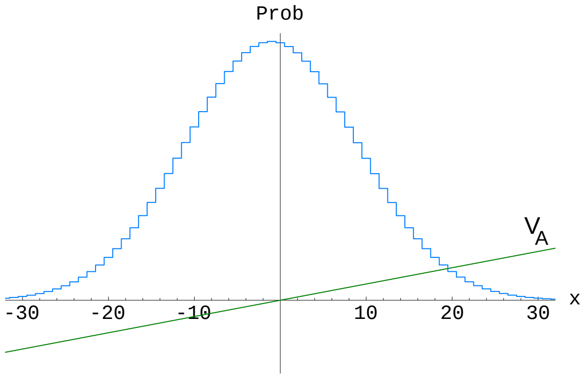

Figure 2. The payoff distribution for game A after 100 plays. $V_A(x)$ is also graphed, in different vertical units. The initial distribution concentrated at $x = 0$ has spread and shifted downhill; the peak is now at -1.

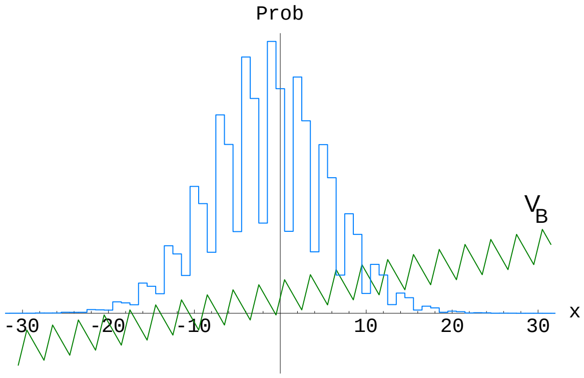

Figure 3. The payoff distribution for game B after 100 plays. $V_B(x)$ is also graphed, in different vertical units. The initial distribution concentrated at $x = 0$ has spread and concentrated in the valleys of V_B, but also shifted downhill.

where $c = (2p_a - 1)\Delta x/\Delta t$. Equation (5) describes Brownian motion of a particle in a linear potential $V_A(x) \propto -(2p_a - 1)x$; the particle diffuses and tends downhill, as shown in Figure 2* for the case $p_a = \frac{1}{2} - \epsilon$ simulated in §1.

Similarly, game B corresponds to Brownian motion of a particle in a piecewise linear potential. For a fair game, *i.e.*, for p_0 and p_1 satisfying equation (3), the potential (as well as its gradient) is periodic:

$$V(x) \propto \begin{cases} -(2p_0 - 1)x & \text{if } |x - 3n| \le b,\ n \in \mathbb{Z}; \\ -(2p_1 - 1)x & \text{otherwise.} \end{cases} \tag{6}$$

Here we assume $0 \le p_0 < \frac{1}{2} < p_1 < \min\{1, (3 - 4p_0)/2\}$ and hence

$$0 < b = \frac{3(2p_1 - 1)}{4(p_1 - p_0)} < 1$$

makes the piecewise linear potential *continuous*. For the losing game B simulated in §1, subtracting ϵ from the fair game probabilities $\frac{1}{10}$ and $\frac{3}{4}$ for p_0 and p_1 corresponds to adding the A game potential to the fair B game potential of (6): $V_B(x) = V(x) + V_A(x)$. In this potential, as shown in Figure 3*, the particle diffuses, concentrates in valleys, and tends downhill.

* Figures 2–4 correspond to the same *exact* calculation of the distributions of payoffs for which the expectation values are plotted in Figure 1. To compensate for the familiar $\mathbb{Z}_2$ 'spurious' conserved quantity in $1+1$ dimensional LGA [13], the '$t = 100$' distributions plotted in Figures 2–4 are actually $[p(x, 99) + 2p(x, 100) + p(x, 101)]/4$.

Finally, Figure 4* shows the distribution of payoffs for the combined $AABB$ game. Alternating the games models a 'flashing' potential [14], which allows diffusion uphill during A to be concentrated into uphill valleys by B, leading to an average movement *uphill*. This phenomenon illustrates the use of a ratchet as a thermal engine, first explained by Smoluchowski [15] and subsequently discussed by Feynman [16], by Parrondo and Español [17], and by Abbott, Davis and Parrondo [18]. Such Brownian ratchets have been created experimentally in electromechanical [19] and optical [20] systems.

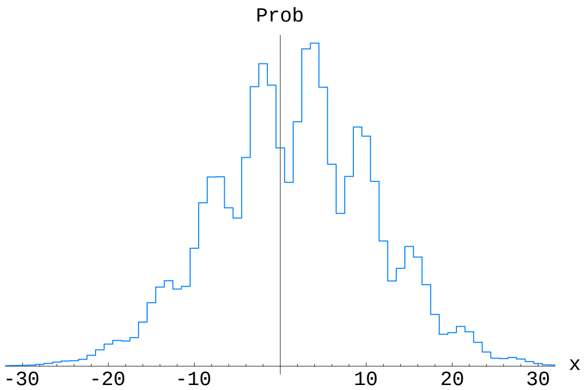

Figure 4. The payoff distribution for the alternating game $AABB$ after 100 plays. Although the initial distribution has spread and concentrated, it has shifted *uphill*.

Recognizing Parrondo games as Brownian ratchets raises concerns about constructing quantum mechanical versions of them [9]: the thermal ratchet engine works only for systems which are out-of-equilibrium (they require heat baths at two different temperatures) and dissipative (the pawl must bounce inelastically off the ratchet). It is hard to imagine exactly unitary systems modelling either of these properties. In fact, recent theoretical analysis [21] and experimental observation [22] of quantum ratcheting have depended on some degree of dissipation/decoherence. Our goal, in contrast, is an exactly *unitary* model.

3. Correlated random walks

The first obstacle we must overcome is the non-existence of a quantum random walk. More precisely, there is no nontrivial unitary band diagonal matrix which would describe the transition amplitudes from each lattice site to some neighboring set of lattice sites [23]. The intuition for this result is that the evolution of nontrivial classical random walks is not invertible and unitarity is simply the quantum manifestation of invertibility.

To construct an invertible model we must add an extra bit of information to each lattice site in $\mathbb{Z}$, the direction from which the particle reached that site. Figure 5 illustrates such a model: the arrows pointing to lattice sites record the direction from which the site was reached and the (probabilistic) evolution rule shown is that the particle has probability p of continuing in the same direction and probability $1-p$ of changing direction. This is a *correlated* random walk [10]: the probabilities for successive steps are not independent for

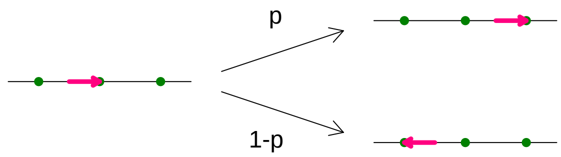

Figure 5. Evolution rules for a correlated random walk. The reflected rules may or may not have the same probabilities; if they do not, the random walk is biased.

$p \neq \frac{1}{2}$. For $p = \frac{1}{2}$, however, they are uncorrelated, so this model specializes to the standard random walk. To obtain an uncorrelated but biased random walk, the probabilities should be independent of the previous outcome, but not symmetric under reflection (*i.e.*, parity).

We can also think of this as a probabilistic LGA. The extra bit of information is the particle momentum and, as we have described it, one timestep of the evolution consists of two parts: scattering, defined by a stochastic matrix

$$S = \begin{array}{c c} & \begin{array}{cc} \leftarrow & \rightarrow \end{array} \\ \begin{array}{c} \leftarrow \\ \rightarrow \end{array} & \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix} \end{array}, \tag{7}$$

followed by advection. Although this is the opposite order to the usual way we think of LGA evolution, the two only differ by a time translation of 'half a timestep'. In fact, long before the earliest LGA were constructed to model fluid flow [24], Goldstein [25] and Kac [26] showed that this probabilistic LGA is a discrete model for a physical system—a $1+1$ dimensional wave equation with dissipation:

$$\frac{1}{v^2}\phi_{tt} + \frac{2a}{v^2}\phi_t - \phi_{xx} = 0,$$

where $v = \Delta x/\Delta t$ and $a = (1-p)/\Delta t$. The $a \to 0$ limit of this 'telegrapher equation' is the wave equation, and the $a, v \to \infty$ limit with $v^2/2a = D$ is the diffusion equation (4).

This correlated random walk/probabilistic LGA corresponds to a generalization of coin flipping games in which the probability of winning each play depends on the outcome of the previous play, and thus provides a framework in which to generalize Parrondo games. Parrondo, Harmer and Abbott have also introduced a generalization in which the probability of winning each play depends on the history of the game to that point—the past two outcomes in their case—although their motivation is to eliminate the x dependence of the game [27]. Our motivation is different: we want to preserve this dependence, since it corresponds to a spatially varying potential, but use the generalization instead to construct unitary versions of these games.

4. Quantum lattice gas automata

Now that we have a *stochastic* scattering matrix (7), it is straightforward to replace it with a *unitary* matrix

$$U = \begin{array}{c c} & \begin{array}{cc} \leftarrow & \rightarrow \end{array} \\ \begin{array}{c} \leftarrow \\ \rightarrow \end{array} & \begin{pmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{pmatrix} \end{array}, \tag{8}$$

although we must reinterpret the state space of the LGA to do so. Let $|x, \alpha\rangle$ denote the presence of a particle at lattice site $x \in \mathbb{Z}$ with momentum $\alpha \in \{\pm 1\}$. States of the probabilistic LGA are convex combinations

$$f = \sum f_{x,\alpha}|x, \alpha\rangle, \quad \text{with } 0 \leq f_{x,\alpha} \in \mathbb{R} \text{ and } \sum f_{x,\alpha} = 1, \tag{9}$$

so that $f_{x,\alpha}$ is the probability that the particle is in the state $|x,\alpha\rangle$. Evolution consists of scattering according to (7):

$$|x,\alpha\rangle \mapsto p|x,\alpha\rangle + (1-p)|x,-\alpha\rangle,$$

followed by advection

$$\mapsto p|x+\alpha,\alpha\rangle + (1-p)|x-\alpha,-\alpha\rangle,$$

extended by linearity to general states f (9).

For a QLGA, the general one particle state is a vector in Hilbert space [2,4,5,28]:

$$\psi = \sum \psi_{x,\alpha}|x,\alpha\rangle, \quad \text{with } \psi_{x,\alpha} \in \mathbb{C} \text{ and } \sum |\psi_{x,\alpha}|^2 = 1, \tag{10}$$

so that $\psi_{x,\alpha}$ is the *amplitude* of the state $|x,\alpha\rangle$ and $|\psi_{x,\alpha}|^2$ is the probability that, if measured in this basis, the particle is observed to be in state $|x,\alpha\rangle$. Quantum evolution consists of scattering according to (8):

$$|x,\alpha\rangle \mapsto \cos\theta|x,\alpha\rangle + i\sin\theta|x,-\alpha\rangle,$$

followed by advection

$$\mapsto \cos\theta|x+\alpha,\alpha\rangle + i\sin\theta|x-\alpha,-\alpha\rangle,$$

extended by linearity to general states ψ (10). This evolution is unitary because the scattering stage is, and because the advection is deterministic. Furthermore, we can include a potential with multiplication by an x-dependent phase $e^{-iV(x)}$ [4,28]; the evolution remains unitary. The problem thus reduces to picking parameters θ, $V(x,t)$ to achieve ratcheting—which we can also interpret as a quantum Parrondo phenomenon.

5. Quantum Parrondo games

Since we are going to exhibit our results as outputs of simulations, we should first remark that although we may think of our single particle QLGA as a particle moving from lattice site to lattice site with specified amplitudes, on a classical computer we must simulate it using a lattice Boltzmann method. That is, we must keep track of the whole vector ψ and evolve that at each timestep. In fact, this is how we performed the exact computations for the probabilistic LGA for Figures 1–4. In the probabilistic case we have the option of simulating it as a lattice gas and averaging over multiple runs—the results of Harmer and Abbott were obtained this way, using 50,000 runs [7]—but for the quantum case we do not have this option.

We set $\theta = \frac{\pi}{4}$ in (8) so that the magnitudes of the amplitudes are all the same—this is the analogue of an unbiased, uncorrelated random walk. The initial state is an equal superposition of $|0,-1\rangle$ and $|0,+1\rangle$ so that there is the same initial capital—zero—as in the classical simulations, and no bias for the initial momentum/state at $t=-1$. Figure 6 shows the expectation value $\langle x\rangle$ as a function of t for

$$V_A(x) = \frac{2\pi}{5000}x \quad \text{and} \quad V_B(x) = \frac{\pi}{3}[1 - \frac{1}{2}(x \bmod 3)] + V_A(x).$$

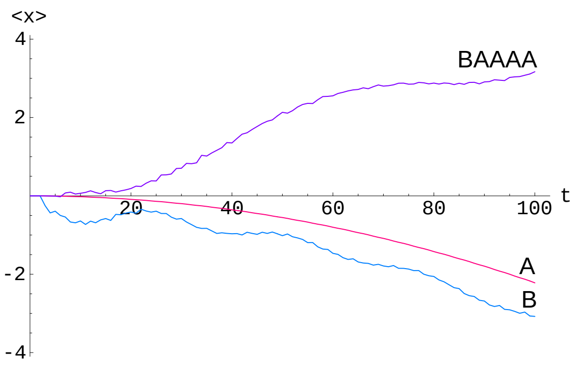

Figure 6. The expected payoffs for quantum games B, A and $BAAAA$ as a function of number of plays t. Although A and B are losing games, the combination $BAAAA$, played repeatedly, is a winning game over this range of numbers of plays. These results illustrate the same 'paradoxical' phenomenon as those in Figure 1 do for the classical Parrondo game.

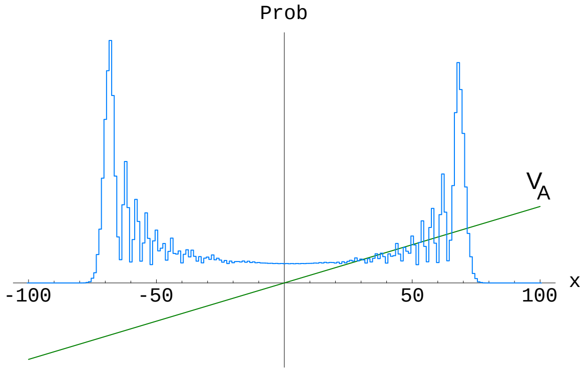

Figure 7. The payoff distribution for quantum game A after 100 plays. $V_A(x)$ is also graphed, in different vertical units. The initial distribution concentrated at $x = 0$ contained equal left and right moving amplitudes which have shifted to peaks at about ± 68 and spread. Interference has created a series of peaks at smaller absolute payoffs; the average has shifted slightly downhill.

As in the classical case, V_A is a linear potential, as shown in Figure 7, and V_B is a piecewise linear 3-periodic potential superimposed on V_A, as shown in Figure 8. Figure 6 shows that the behavior is similar to the classical cases: Potentials V_A and V_B individually force $\langle x \rangle$ downhill, but the flashing pattern—$BAAAA$, repeated—drives $\langle x \rangle$ *uphill*. (We chose the parameters in these potentials to produce expectation value curves similar to those shown in the classical cases; they differ by only about a factor of 2 after 100 plays.) As shown in Figure 7, the evolution in V_A is biased downhill, but looks very little like the diffusive evolution of Figure 2. Similarly, as shown in Figure 8, the evolution in V_B is biased downhill, and concentrates periodically, but otherwise looks quite different than the

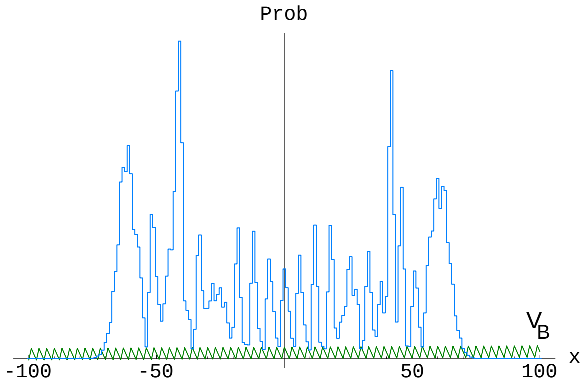

Figure 8. The payoff distribution for quantum game B after 100 plays. $V_B(x)$ is also graphed, in different vertical units. The initial distribution has shifted left/right, and spread. V_B has caused a more complicated interference pattern than V_A, but the average has also shifted downhill.

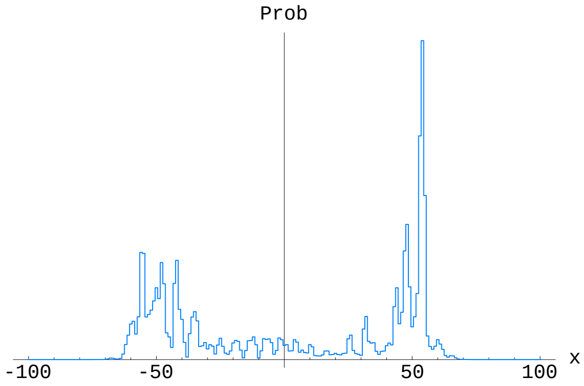

Figure 9. The payoff distribution for the alternating game $BAAAA$ after 100 plays. The distribution still shows the results of interference, but the large positive peak slightly outweighs the large negative peak to give an average which has moved *uphill*.

diffusive evolution of Figure 3. Finally, as shown in Figure 9, flashing the potentials in the order $BAAAA$, repeatedly, biases the evolution uphill, but still in a way unlike the classical case of Figure 4. Interpreting this QLGA as a quantum Parrondo game, Figure 9 shows that this is a game for gamblers with high tolerance for risk—the large probability of a big loss is just barely outweighed by the slightly larger probability of a big win.

6. Conclusions

By interpreting classical Parrondo games as probabilistic LGA, we have motivated the introduction of QLGA to answer the question: Are there quantum Parrondo games? The simulations shown in §5 appear to answer this question in the affirmative, as well as to demonstrate discrete quantum ratcheting, despite the absence of dissipation.

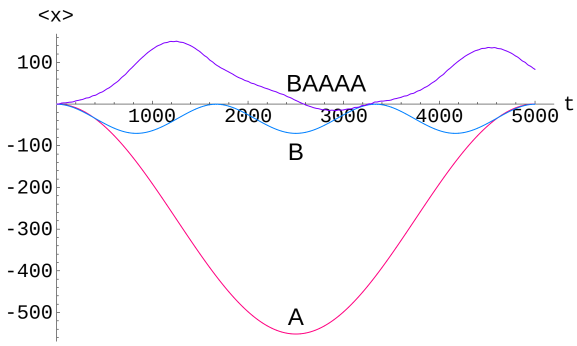

Figure 10. The expected payoffs for quantum games A, B and $BAAAA$ as a function of number of plays t. Although the curves are periodic, for random times (or on average), A and B have negative expected payoffs while $BAAAA$ has a positive expected payoff.

The quadratic growths of $\langle x\rangle$ shown in Figure 6, however, should be worrisome since the single particle QLGA discretizes the Dirac equation [2,4,28], which is relativistic. If $\langle x\rangle$ were to continue to grow quadratically, it would eventually exit the lightcone—not relativistic behavior. Figure 10 shows the results of simulation out to $t = 5000$. We see that the expectation values do not continue to grow quadratically; rather their evolution is oscillatory and the small t quadratic growth is that of $A(\cos(bt)-1)$. In fact, the QLGA with potential V_A discretizes the 'Dirac oscillator' [29] which can be solved exactly, and in which wave packets are known to evolve approximately periodically [30]. For random stopping times—*i.e.*, measurement times—however, both the A and B games are losing games and the $BAAAA$ quantum game is a winning game. In this sense the QLGA is a quantum Parrondo game. In the broader context of game theory, it also illustrates a coherently repeated quantum game—and the possibility of interference between sequences of plays. This kind of construction should generalize to, for example, a quantum version [31] of the MINORITY game [32].

More physically, Aharonov, Ambainis, Kempe and Vazirani also use random stopping times to obtain a related result: quantum (unitary) simulation of sampling from equilibrium distributions of diffusion processes on graphs with constant vertex degree [33]. The quadratic speedup they find is due to the linear in time (rather than $\sqrt{t}$ as in the classical random walk) spread of the wave function illustrated in Figure 7 [34]. More generally, Childs, Farhi and Gutmann demonstrate the same quadratic speedup for a *continuous* time quantum process on certain graphs, without the constant vertex degree restriction [35]. Our results, and these, provide specific answers to the general question of whether quantum computers (see, *e.g.*, [36]) can calculate properties of classical systems more efficiently than can classical computers [37].

Acknowledgements

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