

# SCALE DECOHERENCE IN INHOMOGENEOUS POTENTIALS

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## ABSTRACT

Finite precision measurement factors the Hilbert space of a quantum system into a tensor product  $H_{\text{coarse}} \otimes H_{\text{fine}}$ . This is mathematically equivalent to the partition into system and environment which forms the arena for decoherence, so we describe the consequences of the inaccessibility of  $H_{\text{fine}}$  as *scale decoherence*. Considering the experimentally important case of a harmonic oscillator potential as well as a periodic piecewise constant potential, we show that scale decoherence occurs for inhomogeneous potentials and may explain part of the decoherence observed in recent and proposed experiments on mesoscopic superpositions of quantum states.

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Mesoscopic superpositions of quantum states were prepared [1,2] and observed to decohere [2] in remarkable experiments reported recently. These experimental successes coincided with calls for renewed theoretical investigation of decoherence [3,4] and proposals for experimental measurement of the density matrix [1,5] of mesoscopic quantum systems. Among the motivations for much of this work is the prospect of quantum computing [6]: Not only does the power of quantum algorithms depend on quantum superpositions [7,8], but both the ion trap [1] and cavity QED [2] systems used in these experiments have been proposed for the physical realization of Boolean quantum computers [9]. Even these relatively ‘clean’ realizations of quantum computation, however, will have runtimes limited by decoherence [10]. Current proposals for quantum error correction [11] and fault tolerant quantum computation [12] aim to circumvent such limits; detailed theoretical and experimental understanding of the decoherence of quantum states may indicate how to implement these proposals successfully.

Careful analysis of the complexity of quantum algorithms considers only *finite precision* specification of transition amplitudes [8], reminding us that only finite precision measurements for quantum state observables with continuous spectra are physically realistic. In this paper we consider finite precision measurement of the position of a quantum particle evolving according to the Schrödinger equation and demonstrate that in the presence of an inhomogeneous potential the *measured* quantum state generically decoheres— independently of any coupling to an environment—a phenomenon which we will call *scale decoherence*. The recognition that quantum measurements have only finite precision/limited resolution dates back at least to von Neumann [13]. Coarse graining of various sorts, including by finite precision measurement [14], has more recently been used in the consistent histories approach to quantum mechanics to obtain decoherent sets of histories [15]. The scale decoherence we exhibit here in the reduced density matrix is the Schrödinger picture version of this phenomenon and should be apparent in experimental measurements of quantum systems having inhomogeneous potentials. These are common in the experimental arrangements used for precision measurements of quantum systems [3]; the Paul trap used in [1], for example, constrains the ions within a harmonic oscillator potential.

Our model for the limited resolution of experimental measurements is the simplest possible [13]: the continuous spectrum of the operator of interest is partitioned into congruent ‘bins’ [16] corresponding to possible finite precision readings of a measuring device. For the one dimensional position measurements with which we are concerned here, we take the bins to be intervals of length  $\epsilon > 0$  and, in keeping with the computer science connotations of ‘finite precision’, write  $x = y\epsilon + z$  for a position  $x \in \mathbb{R}$ , with  $y \in \mathbb{Z}$  labelling the bin and  $0 \leq z < \epsilon$  the position inside the bin. Resolving the mathematical difficulties in the usual way [17], we take the set of position eigenstates  $\{|x\rangle\}$  as a basis for the Hilbert space  $H$  of one particle wave functions. Now notice that writing  $|x\rangle = |y\epsilon + z\rangle = |y\rangle \otimes |z\rangle$  defines a tensor product decomposition  $H = H_{\text{coarse}} \otimes H_{\text{fine}}$ . Since we are taking the  $H_{\text{fine}}$  factor of the Hilbert space to be inaccessible to measurement, the situation is mathematically equivalent to the decomposition of Hilbert space into system and environment factors usual in discussions of decoherence (whether it is conceptualized as damping [18],

measurement [12,19,20], relaxation [21] or dissipation [22]): small scale position functions as an unmeasured ‘environment’ ‘coupled’ to the finite precision position ‘system’.

An arbitrary vector  $|\psi\rangle \in H$  need not, of course, factor as a tensor product of vectors in  $H_{\text{coarse}}$  and  $H_{\text{fine}}$ . Any quantum state represented as a density operator  $\rho \in H \otimes H^*$ , however, can be traced over  $H_{\text{fine}}$  to obtain a *scale  $\epsilon$  reduced density operator*  $\tilde{\rho} := \text{Tr}_{\text{fine}}\rho \in H_{\text{coarse}} \otimes H_{\text{coarse}}^*$ , describing the quantum state as measurable with finite precision. The trace over  $H_{\text{fine}}$  is exactly an integral over small scales: for a pure state  $|\psi\rangle \in H$ , the density function is  $\rho(x, x') := \psi(x)\psi(x')$  and the scale  $\epsilon$  reduced density function is

$$\tilde{\rho}(y, y') := \int_0^\epsilon \psi(y\epsilon + z)\overline{\psi(y'\epsilon + z)} dz \quad (1)$$

for  $y, y' \in \mathbb{Z}$ . Notice that plane waves are tensor products relative to the decomposition of  $H$  just as are the position eigenstates. In this case the scale  $\epsilon$  reduced density operator has rank 1:

$$\tilde{\rho}(y, y') = \int_0^\epsilon e^{ik(y\epsilon+z)} e^{-ik(y'\epsilon+z)} dz =: e^{ik(y-y')\epsilon} I(0),$$

where  $I(0) = \epsilon$ ; and thus the *scale  $\epsilon$  entropy*  $S := -\text{Tr}\tilde{\rho}\log\tilde{\rho}$  vanishes as it does for the original pure state.

Now consider a superposition of plane waves:  $\psi(x) = a_1 e^{ik_1 x} + a_2 e^{ik_2 x}$ . The scale  $\epsilon$  reduced density function (1) is:

$$\begin{aligned} \tilde{\rho}(y, y') &= a_1 \bar{a}_1 e^{ik_1(y-y')\epsilon} I(0) + a_1 \bar{a}_2 e^{i(k_1 y - k_2 y')\epsilon} I(k_1 - k_2) \\ &\quad + a_2 \bar{a}_1 e^{i(k_2 y - k_1 y')\epsilon} I(k_2 - k_1) + a_2 \bar{a}_2 e^{ik_2(y-y')\epsilon} I(0), \end{aligned}$$

where

$$I(k_1 - k_2) := \int_0^\epsilon e^{i(k_1 - k_2)z} dz. \quad (2)$$

It is clear that in this case, for  $k_1 \neq k_2$ ,  $\tilde{\rho}$  has rank 2. In fact,  $\tilde{\rho}$  is easy to diagonalize: Its eigenvectors must be linear combinations  $\phi(y') = b_1 e^{ik_1 y' \epsilon} + b_2 e^{ik_2 y' \epsilon}$ ; setting

$$\sum_{y'} \tilde{\rho}(y, y') \phi(y') = \lambda \phi(y)$$

we find that for  $\phi$  to be an eigenvector  $\lambda$  must be a root of the characteristic equation

$$0 = \det \begin{pmatrix} a_1 \bar{a}_1 - \lambda & a_1 \bar{a}_2 I(k_1 - k_2)/I(0) \\ a_2 \bar{a}_1 I(k_2 - k_1)/I(0) & a_2 \bar{a}_2 - \lambda \end{pmatrix}. \quad (3)$$

Now the entropy of the scale  $\epsilon$  reduced density matrix does not vanish: we can solve (3) for  $\lambda$  and examine the dependence of the eigenvalues (and thence the entropy) on  $\epsilon$ . From (2) it is clear that the relevant quantity is  $|k_1 - k_2|\epsilon$ : when  $|k_1 - k_2|\epsilon \ll 1$  the scale  $\epsilon$  entropy almost vanishes; only when  $\epsilon$  is at least comparable to  $|k_1 - k_2|^{-1}$ , which we can consider

to be the characteristic spatial scale of  $\psi(x)$ , is there substantial entropy. As  $\epsilon \rightarrow \infty$  the scale  $\epsilon$  entropy (in bits) saturates at 1.

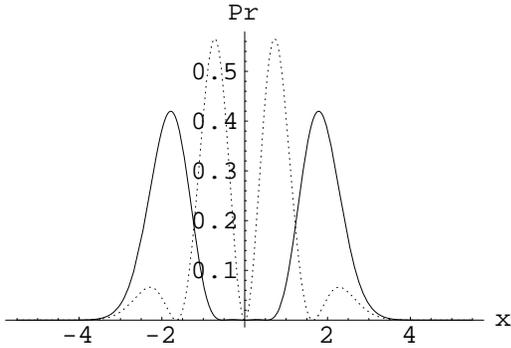
Notice, furthermore, that the eigenvalues  $\lambda$  depend on the Fourier coefficients  $a_n$  in the superposition only through their squared norms  $a_n \bar{a}_n$ . In fact, this is true for superpositions of any number of plane waves. For a homogeneous potential plane waves are stationary states, so since the time evolution of each is by a phase  $e^{-i\omega_n t}$ , the scale  $\epsilon$  entropy, even if it is nonzero, must be constant in time; no initial state scale decoheres.

For an inhomogeneous potential the situation is more interesting: the eigenstates are still stationary, but they are not plane waves. Consequently the scale  $\epsilon$  reduced density operator for a wave function expressed as a superposition of eigenstates is more complicated to diagonalize. Generically the eigenvalues depend on the phases of the superposition coefficients as well as their squared norms and hence the scale  $\epsilon$  entropy is time dependent.

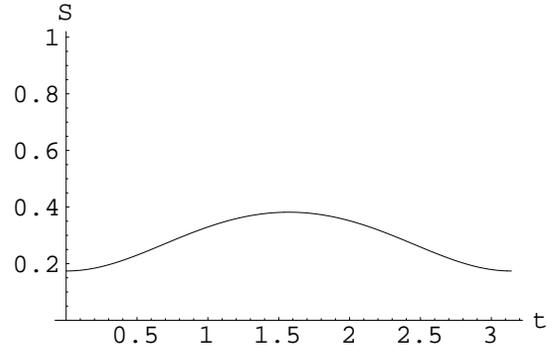
Let us examine the case of the harmonic oscillator potential which is present in the experimental arrangements described in [1,23,24] and proposed in [5], for example. An initial eigenstate does not decohere, of course, since the density operator in this case is time independent. So consider an initial superposition of two eigenstates  $u_i$  of the harmonic oscillator Schrödinger operator, as was prepared in the experiments reported in [23,24]:  $\psi(x) = (u_1(x) - u_3(x))/\sqrt{2}$ , say. The scale  $\epsilon$  reduced density function  $\tilde{\rho}(y, y')$  for such an superposition is a complicated and not particularly enlightening expression involving the error function,  $\text{Erf}(\cdot)$ . More informative is direct numerical evaluation of the eigenvalues and the scale  $\epsilon$  entropy as a function of time.

To do so we normalize the potential to  $V(x) = x^2/2$  and calculate the reduced density matrix for  $\epsilon = 1/2$ . (In order to preserve the symmetry of the problem we shift the domain of integration in (1) to  $[-\epsilon/2, \epsilon/2]$ .) For numerical purposes the wave function vanishes already for  $|x|$  greater than about 5 as is indicated by the probability distributions plotted in Figure 1; although  $\epsilon = 1/2$  is a significant fraction of the size of the support of the wave function it is still small relative to the characteristic spatial scales, approximately 4 and 2, respectively, of  $u_1$  and  $u_3$ , so we do not expect the scale  $\epsilon$  entropy to be saturated.

Figure 2 shows the scale  $\epsilon$  entropy (in bits) computed for the time evolution of the initial superposition. The scale  $\epsilon$  entropy initially increases—the state as measured with finite precision decoheres. In the longer run, however, the scale  $\epsilon$  entropy is oscillatory. In fact, since the eigenstates in the superposition have periods  $4\pi/3$  and  $4\pi/7$ , respectively, the evolution must be exactly periodic with period  $\pi$ , as shown in Figure 2. This type of oscillatory behavior for the entropy is familiar from studies of the decoherence of a finite dimensional quantum system coupled to a finite dimensional environment, *e.g.*, one spin coupled to another [20,4]. Here the ‘environment’  $H_{\text{fine}}$  is infinite dimensional, but only a small finite dimensional subspace—determined by the Fourier components of the eigenstates superposed in the initial state—has nonnegligible ‘coupling’ to the essentially finite dimensional ‘system’  $H_{\text{coarse}}$ . This means, in addition, that the scale  $\epsilon$  entropy is bounded above. The maxima of the entropy curve occur at times when the wave function



**Figure 1.** The position probability distributions computed from the wave function at time 0 (solid curve) and time  $\pi/2$  (dotted curve).



**Figure 2.** Entropy as a function of time for the scale  $\epsilon$  reduced density matrix of an initial superposition of harmonic oscillator eigenstates.

has the smallest characteristic spatial scale. Figure 1 shows  $|\psi(x, t)|^2$  for  $t = 0$  and  $t = \pi/2$ , at which times the scale  $\epsilon$  entropy is minimal and maximal, respectively.

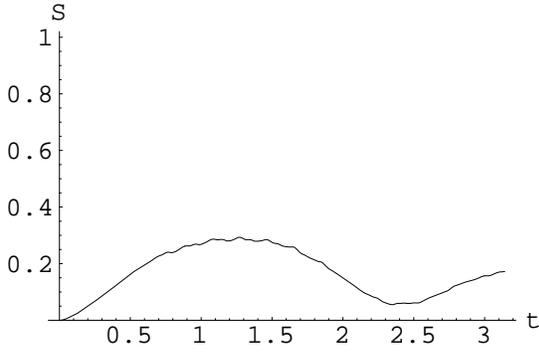
The minima of the entropy curve in Figure 2 do not give vanishing entropy; that the entropy does not vanish at  $t = 0$  (and would be even larger than it is were  $\epsilon > 1/2$ ) is uncharacteristic of typical analyses of decoherence where the initial state is often taken to factor relative to the tensor product decomposition of the Hilbert space [18–22,3,25]. To illustrate this situation for scale decoherence let us consider the case of a periodic potential where we can naturally take the initial state to be a plane wave which, as we noted earlier, is a tensor product relative to the scale decomposition of  $H$ .

Specifically, we consider the Schrödinger equation defined on the circle  $S^1 = \mathbb{R}/2\pi\mathbb{Z}$  with the potential

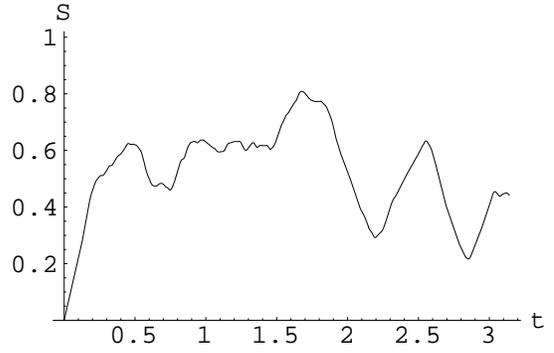
$$V(x) = \begin{cases} V_0 & -\pi/2 \leq x < \pi/2 \pmod{2\pi}; \\ 0 & \text{otherwise.} \end{cases} \quad (4)$$

An eigenstate of energy  $E$  is, of course, a linear combination of  $e^{i\sqrt{E}x}$  and  $e^{-i\sqrt{E}x}$  where  $V(x) = 0$  and a linear combination of  $e^{i\sqrt{E-V_0}x}$  and  $e^{-i\sqrt{E-V_0}x}$  where  $V(x) = V_0$ . The usual requirement that the wave function and its first derivative be continuous determines the ratios of the coefficients in these superpositions. Requiring the wave function to be periodic makes the spectrum of eigenenergies discrete: for  $V_0 = 0$ ,  $E_n = n^2$  for  $n \in \mathbb{Z}_{\geq 0}$ ; for  $V_0 > 0$  the eigenvalues  $E_n^\pm$  split (for  $n > 0$ ) and are roots of a transcendental equation. Let  $\psi_n^\pm$  (and  $\psi_0^+$ ) denote the corresponding symmetric/antisymmetric eigenstates.

When  $V_0 > 0$  a plane wave  $e^{ikx}$  for  $k \in \mathbb{Z}$ , although periodic, is not an eigenstate. Rather it is a superposition of eigenstates with dominant contributions coming from ones with energies near  $k^2$  and near  $k^2 + V_0$ . Just as in our earlier example of a superposition of two harmonic oscillator eigenstates, this means that the initial plane wave state will decohere. To make the scales comparable, consider the case  $k = 1$  in the potential (4) with  $V_0 = 3$ . We expect the dominant components of the superposition to be eigenstates with



**Figure 3.** Entropy as a function of time for the scale  $\epsilon$  reduced density matrix of an initial  $k = 1$  plane wave in the potential (4) with  $V_0 = 3$ .



**Figure 4.** Entropy as a function of time for the scale  $\epsilon$  reduced density matrix of an initial  $k = 1$  plane wave in the potential (4) with  $V_0 = 15$ .

energies near 1 and near 4. In fact,

$$e^{ix} \approx -0.41\psi_0^+ + 0.55\psi_1^+ + 0.66i\psi_1^- + 0.18\psi_2^+ + 0.24\psi_2^- \\ + 0.01\psi_3^+ + 0.03\psi_3^- - 0.02\psi_4^-, \quad (5)$$

where all components with amplitudes of norm at least 0.0001 are included and the energies of the first 5 eigenstates listed are approximately 0.52, 3.27, 2.01, 5.41 and 5.84, respectively, all of which are reasonably close to 1 and/or 4. The corresponding characteristic spatial scales are near  $2\pi$  and  $\pi$ , respectively, so we set the precision  $\epsilon = \pi/4$  to be comparable to the  $\epsilon = 1/2$  precision we used in the harmonic oscillator calculation.

Figure 3 shows the scale  $\epsilon$  entropy (in bits) for the  $k = 1$  initial plane wave (5). Since this initial state does factor as a tensor product relative to the scale decomposition of  $H$ , the scale  $\epsilon$  entropy vanishes at  $t = 0$ . Then, just as in Figure 2, the scale  $\epsilon$  entropy increases (to a similar maximum value) before decreasing. In this case, however, the multiple higher frequency components in the original superposition manifest themselves in the irregularity of the entropy curve and prevent the entropy from decreasing all the way to its initial (zero) value before it again increases.

Measuring position with greater precision, *i.e.*, decreasing  $\epsilon$ , reduces the amount of scale  $\epsilon$  decoherence. On the other hand, increasing the amplitudes of higher frequency components amplifies the scale  $\epsilon$  decoherence. Figure 4 shows the scale  $\epsilon$  entropy for the potential (4) with  $V_0 = 15$  but still with  $\epsilon = \pi/4$  and initial state a plane wave with  $k = 1$  as in Figure 3. Now the dominant components are eigenstates with energies near 1 and 16, and with characteristic spatial scales near  $2\pi$  and  $\pi/2$ , respectively. Including only components of the same relevance as in (5) we have now

$$e^{ix} \approx -0.49\psi_0^+ + 0.11\psi_1^+ + 0.54i\psi_1^- + 0.44\psi_2^+ + 0.29i\psi_2^- \\ + 0.25\psi_3^+ + 0.36\psi_3^- - 0.01\psi_4^- - 0.01\psi_5^+ - 0.06\psi_5^- \\ + 0.02\psi_6^- + 0.02\psi_7^- - 0.01\psi_8^-, \quad (6)$$

where the energies of the first 7 eigenstates listed are approximately 0.74, 6.47, 2.92, 15.28, 11.15, 17.49 and 18.47, respectively, all of which are near the range [1, 16]. Now the scale  $\epsilon$  entropy increases even more rapidly from 0 than in Figure 3, and the presence of additional high frequency components makes the entropy curve so irregular as to obscure *any* approximate periodicity in time. The common period of the eigenstates with appreciable amplitudes in (6) is so large that the initial state undergoes essentially irreversible scale  $\epsilon$  decoherence.

These last two computations demonstrate that initial tensor product states scale decohere with the same ‘ultraviolet burst’ that is familiar from other decoherence calculations [3]. Moreover, the last computation shows that when the finite precision ‘system’ ‘couples’ strongly to multiple dimensions of the small scale ‘environment’, scale decoherence also approaches the irreversible behavior of decoherence by an infinite dimensional thermal environment [18–22].

Most importantly, we have shown that scale decoherence occurs whenever the potential is inhomogeneous, which includes the experimentally relevant situation of a harmonic oscillator,\* and is mathematically identical to any other form of decoherence. Thus scale decoherence is a real experimental effect for finite precision measurements. Although the measurements described in [1] are not direct position measurements, the demonstration of mesoscopic superpositions depends on the measurement of spin down probability as a function of phase separation  $\phi \in [-\pi, \pi)$ . At least part of the unexplained decoherence [24] in the measurements for initially superposed harmonic oscillator coherent states  $|\alpha\rangle$  and  $|-\alpha\rangle$  with  $\alpha \approx 2.97(6)$  may be scale decoherence due to finite precision measurement of the relative phase angle  $\phi$ —this is consistent with the recognized fluctuations in  $\phi$  [1] and the finer scale variation of the signal (as a function of  $\phi$ ) in this case than for smaller values of  $\alpha$ . Similarly, we may expect scale decoherence in the larger  $\alpha$  cavity QED experiments proposed in [2].

Finally, we remark that scale decoherence affects the measurement and phase space tomographic reconstruction of the density matrix (or equivalently, the Wigner function) for a quantum state [5,23,24]. While not originally conceptualized as decoherence, it is an old result [26], revisited in the consistent histories program [27], that smearing the Wigner function over a suitable volume of phase space produces a positive probability distribution. Finite precision measurement effects exactly such a smearing, although for sufficiently small  $\epsilon$  the resulting scale  $\epsilon$  decoherence does not eliminate all negative values of the Wigner function—and the limited precision need not originate in the quantum mechanical uncertainty of simultaneous position and momentum measurements. Since reconstruction of the density matrix/Wigner function is now often performed in quantum optics [28] as well as in atomic physics, it may prove useful to consider optical homodyne/phase space tomography based on incomplete data [29] from the perspective of scale decoherence.

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\* Eigenstates, as we have seen, do not scale decohere. Nor, in a harmonic oscillator potential, does a coherent gaussian wave packet since evolution preserves its shape; a superposition of such packets will, however, scale decohere.

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