§3.5. Problem 112. Show that there are infinitely many primes of the form $4n + 3$.

Recall that when $p = 4n + 3$, we write $p \equiv 3 \pmod 4$. Suppose there were only a finite number of primes congruent to 3 modulo 4; call them $p_1, \ldots, p_k$. Let $P = p_1 p_2 \cdots p_k$. We consider two cases: k even and k odd.

When k is even, $P \equiv 3^k \equiv 1 \pmod 4$. Then $P+2 \equiv 3 \pmod 4$. According to Prop. 3.2, $P + 2$ is a product of primes. 2 is not one of them since $P + 2$ is odd. If all the primes that divide $P + 2$ were congruent to 1 modulo 4, then $P + 2$ would also be congruent to 1 modulo 4, so there must be at least one prime dividing $P+2$ that is congruent to 3 modulo 4, say p_i. But then $p_i | P$ and $p_i | P + 2$, so $p_i | (P + 2) - P = 2$, which is a contradiction.

When k is odd, $P \equiv 3^k \equiv 3 \pmod 4$. Then $P + 4 \equiv 3 \pmod 4$. Just as in the previous case we can find some p_j dividing both P and $P + 4$, and hence dividing 4. Again this is a contradiction.

Since k can be neither even nor odd, we must have been wrong to suppose there are only a finite number of primes congruent to 3 modulo 4. Thus there are infinitely many primes of the form $4n + 3$.

Note: To prove that the product of an even (odd) number of integers of the form $4n+3$ is congruent to 1 (3) modulo 4, use induction.