This is the final from the last time I taught this class. We have not covered Chapter 9, so you do not need to be able to do problems 1, 2 and 6.b.

There are 6 problems worth a total of 100 points, and 1 extra credit problem worth 15 points. Make sure that you look on the back of this page.

1. [16 points] Suppose $\langle x_n \rangle \to x$ and for all $n \in \mathbb{N}$, $x_n \neq x$. Prove that

$$\langle \frac{x^n - x_n}{x - x_n} \rangle \to 2x.$$

2. [16 points] Use the Least-Upper-Bound Axiom to prove that if $S \subset \mathbb{R}$ has a lower bound then it has a greatest lower bound.

3. For $n \in \mathbb{N}$, let $[n]$ denote the equivalence class of $n$ modulo $p$, where $p > 1$ is a prime number. Explain your answer to each of the following questions:

   a. [2 points] What is the cardinality of $\mathbb{Z}_p = \{[n] \mid n \in \mathbb{N}\}$?

   b. [4 points] Define $f : \mathbb{N} \to \mathbb{Z}_p$ by $f(n) = [n]$. Is $f$ onto? Is $f$ one-to-one?

   c. [6 points] Define $g : \mathbb{N} \to \mathbb{Z}_p$ by $g(n) = [n^{p-1}]$. Is $g$ onto? Is $g$ one-to-one?

   d. [6 points] Let $q > 1$ be a prime number not equal to $p$. Define $h : \mathbb{Z}_p \to \mathbb{Z}_p$ by $h([n]) = [qn]$. Is $h$ onto? Is $h$ one-to-one?

4. a. [6 points] Solve $x^2 - x \equiv 6 \pmod{7}$.

   b. [12 points] Find all solutions to the system of congruences:

   $$x^2 - x \equiv 6 \pmod{7}$$
   $$x \equiv 1 \pmod{5}.$$

(over)
5. Let $n \in \mathbb{N}$.

a. [5 points] How many solutions are there to the equation $x_1 + x_2 = n$ for $x_1$ and $x_2$ nonnegative integers? Prove that your answer is right.

b. [10 points] Prove that there are $(n + 1)(n + 2)/2$ solutions to the equation $x_1 + x_2 + x_3 = n$ for $x_1$, $x_2$ and $x_3$ nonnegative integers.

6. For $n \in \mathbb{N}$, let $U_n = \{x \in \mathbb{R} \mid 1 - \frac{1}{n} < x < 1 + \frac{1}{n}\}$.

a. [5 points] Find $\bigcap_{n \in \mathbb{N}} U_n$.

b. [10 points] Let $\langle a_n \rangle$ be a sequence with the property that for all $n \in \mathbb{N}$, $a_n \in U_n$. Prove $\langle a_n \rangle \rightarrow 1$.

Extra Credit. [15 points] Prove that the following propositional expression is a contradiction:

\[
(\neg R_1 \lor \neg R_2) \land (\neg B_1 \lor \neg B_2) \land \\
(\neg R_2 \lor \neg R_3) \land (\neg B_2 \lor \neg B_3) \land \\
(\neg R_3 \lor \neg R_1) \land (\neg B_3 \lor \neg B_1) \land \\
(R_1 \lor B_1) \land (R_2 \lor B_2) \land (R_3 \lor B_3)
\]

Your proof does not have to include a complete truth table.